Lecture 13: Game Theory // Nash equilibrium

Mauricio Romero
Examples - Continued
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Cournot - Revisited
Bertrand Competition
Bertrand Competition - Different costs
Bertrand Competition - 3 Firms
Hotelling and Voting Models
Cournot Competition

- $N$ identical firms competing on the same market
Cournot Competition

- $N$ identical firms competing on the same market
- Marginal cost is constant and equal to $c$

Aggregate inverse demand is $p = a - bN\sum_{j=1}^{N}q_j$.

Benefits of firm $j$ are:

$$\Pi_j(q_1, \ldots, q_N) = (a - bN\sum_{i=1}^{N}q_i)q_j - cq_j.$$
Cournot Competition

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Cournot Competition

- The FOC for a given firm is:

\[ a - b \sum_{i=1}^{N} q^i - bq_j - c = 0 \]
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- The symmetric Nash equilibrium is given by

\[ q^* = \frac{a - c}{b(N + 1)} \]
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\[ q^* = \frac{a - c}{b(N + 1)} \]

- Thus

\[ \sum_{j=1}^{N} q^j = \frac{N(a - c)}{b(N + 1)} \]

\[ p = a - N \frac{a - c}{(N + 1)} < a \]

\[ \Pi^j = \frac{(a - c)^2}{b(N + 1)^2} \]
Cournot Competition

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\sum_{j=1}^{N} q^j = \frac{N(a - c)}{b(N + 1)}
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> As \( N \to \infty \) we get close to perfect competition
Cournot Competition

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- As \( N \to \infty \) we get close to perfect competition
- \( N = 1 \) we get the monopoly case
Examples - Continued

Cournot - Revisited
Bertrand Competition
Bertrand Competition - Different costs
Bertrand Competition - 3 Firms
Hotelling and Voting Models
Consider the alternative model in which firms set prices.

In the monopolist’s problem, there was no distinction between a quantity-setting model and a price setting.

In oligopolistic models, this distinction is very important.
Consider two firms with the same marginal constant marginal cost of production and demand is completely inelastic.

Each firm simultaneously chooses a price $p_i \in [0, +\infty)$.

If $p_1, p_2$ are the chosen prices, then the utility functions of firm $i$ is given by:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 
0 & \text{if } p_i > p_{-i}, \\
(p_i - c) \frac{Q(p_i)}{2} & \text{if } p_i = p_{-i}, \\
(p_i - c)Q(p_i) & \text{if } p_i < p_{-i}.
\end{cases}$$
Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing \((MR'(p_i) < 0)\):

\[
R(p_i) = p_i Q(p_i) \quad (1)
\]
\[
MR(p_i) = Q(p_i) + p_i Q'(p_i) \quad (2)
\]
\[
= Q(p_i) (1 + \varepsilon_{Q,p}(p_i)) . \quad (3)
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Bertrand Competition

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- Let \( p^m > c \geq 0 \) be the monopoly price such that \( MR(p^m) = c \).
Bertrand Competition

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\]

Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

Then

\[
MR(p_i) - c > 0 \text{ if } p_i < p^m, \quad MR(p_i) - c < 0 \text{ if } p_i > p^m.
\]
The best response function is:

\[ BR_i(p_{-i}) = \begin{cases} 
  p^m & \text{if } p_{-i} > p^m, \\
  p_{-i} - \varepsilon & \text{if } c < p_{-i} \leq p^m, \\
  [c, +\infty) & \text{if } c = p_{-i} \\
  (c, +\infty) & \text{if } c > p_{-i}. 
\end{cases} \]

Where \( \varepsilon \) is the smallest monetary unit.
Bertrand Competition

Case 1: $p_1^* > p^m$

$\Rightarrow p_2^* = p^m$

So this cannot be a Nash equilibrium
Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$

So this cannot be a Nash equilibrium.
Bertrand Competition

Case 1: \( p_1^* > p^m \)

- \( p_2^* = p^m \)

- \( BR_2(p^m) = p^m - \varepsilon \)

- \( BR_1(p^m - \varepsilon) = p^m - 2\varepsilon \)

So this cannot be a Nash equilibrium.
Bertrand Competition

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Case 2: \( p_1^* \in (c, p^m] \)

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Case 2: \( p_1^* \in (c, p^m] \)

- \( \text{BR}_2(p_1^*) = p_1^* - \varepsilon \)

- \( \text{BR}_1(p_1^* - \varepsilon) = p_1^* - 2\varepsilon \)

- So this cannot be a Nash equilibrium
Case 3: $p_1^* < c$

$BR_2(p_1^*) \in [p_1^* + \epsilon, \infty)$
Bertrand Competition

Case 3: $p_1^* < c$

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- So this cannot be a Nash equilibrium
Case 4: $p_1^* = c$

$BR_2(p_1^*) = (c, +\infty)$
Case 4: \( p_1^* = c \)

- \( BR_2(p_1^*) = (c, +\infty) \)

- The unique pure strategy Nash equilibrium is \( p_1^* = p_2^* = c \)
Bertrand Competition

Thus in contrast to the Cournot duopoly model, in the Bertrand competition model, two firms get us back to perfect competition \((p = c)\)
Examples - Continued

Cournot - Revisited
Bertrand Competition

**Bertrand Competition - Different costs**
Bertrand Competition - 3 Firms
Hotelling and Voting Models
Bertrand Competition - different costs

- Suppose that the marginal cost of firm 1 is equal to $c_1$ and the marginal cost of firm 2 is equal to $c_2$ where $c_1 < c_2$.

- The best response for each firm:

$$BR_i(p_{-i}) = \begin{cases} 
p_i^i & \text{if } p_{-i} > p_i^i, \\
p_{-i} - \varepsilon & \text{if } c_i < p_{-i} \leq p_m^i, \\
[c_i, +\infty) & \text{if } p_{-i} = c_i \\
(p_{-i}, +\infty) & \text{if } p_{-i} < c_i.
\end{cases}$$
Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss.
Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss

- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market

- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut $p_2^*$ and get a positive profit

- Firm 1 would really like to price at some price $p_1^*$ just below the marginal cost of firm 2, but wherever $p_2^*$ is set, Firm 1 would try to increase prices

- No NE because of continuous prices
Bertrand Competition - different costs

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Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
Bertrand Competition - discreet prices

▶ Suppose $c_1 = 0 < c_2 = 10$

▶ Firms can only set integer prices.
Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$

- Firms can only set integer prices.

- Suppose that $(p_1^*, p_2^*)$ is a pure strategy Nash equilibrium...
Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$
Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$

- $p_1^*$ cannot be a best response to $p_2^*$ since by setting $p_1 = p_2^*$ firm 1 would get strictly positive profits
Case 2: \( p_1^* \in \{1, 2, \ldots, 9\} \)

- Best response of firm 2 is to set any price \( p_2^* > p_1^* \)
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- Best response of firm 2 is to set any price $p_2^* > p_1^*$

- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price
Bertrand Competition - discrete prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$

- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price

- The only equilibrium is $(p_1^*, p_1^* + 1)$
Bertrand Competition - discreet prices

Case 3: \( p_1^* = 10 \)

- Best responses of firm 2 is to set any price \( p_2^* \geq p_1^* \)
Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$
Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:
  \[
  \frac{1}{2}(10) = 5 < 9.
  \]

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher.
Bertrand Competition - discreet prices

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- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

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$$\frac{1}{2}(10) = 5 < 9.$$ 

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- $(p_1^*, p_2^*) = (10, 11)$ is a Nash equilibrium
Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$
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- Best response of firm 2 is to set $p_2^* = 11$

- Firm 1 would not be best responding since by setting a price of $p_1 = 10$, it would get strictly positive profits
Case 5: $p_1^* \geq 12$

- Firm 2’s best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$
Bertrand Competition - discreet prices

Case 5: \( p_1^* \geq 12 \)

- Firm 2’s best response is to set either \( p_2^* = p_1^* - 1 \) or \( p_2^* = p_1^* \)

- Firm 1 is not best responding since by lowering the price it can get the whole market.
Examples - Continued

- Cournot - Revisited
- Bertrand Competition
- Bertrand Competition - Different costs
- Bertrand Competition - 3 Firms
- Hotelling and Voting Models
Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms

\[
\text{Best response of firm } i \text{ is given by:} \\
\begin{align*}
BR_1(p_2, p_3) &= \begin{cases} 
  p_m & \text{if } \min\{p_2, p_3\} > p_m, \\
  \min\{p_2, p_3\} - \varepsilon & \text{if } c < \min\{p_2, p_3\} \leq p_m, \\
  (c, +\infty) & \text{if } c > \min\{p_2, p_3\}.
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\end{align*}
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\((c, c, c)\) is indeed a pure strategy Nash equilibrium as in the two firm case.
Bertrand Competition - 3 firms

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Bertrand Competition - 3 firms

- If \((p_1, p_2, p_3)\) was a pure strategy Nash equilibrium, it can never be the case that \(\min\{p_1, p_2, p_3\} < c\)
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- We must have \(\min\{p_1, p_2, p_3\} = c\)

- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to \(c\)? No since that firm would want to raise his price a bit and get strictly better profits.

- There must be at least two firms that set price equal to marginal cost.

- Set of all pure strategy Nash equilibria are given by:
  \[
  \{(c, c, c + \varepsilon) : \varepsilon \geq 0\} \cup \{(c, c + \varepsilon, c) : \varepsilon \geq 0\} \cup \{(c + \varepsilon, c, c) : \varepsilon \geq 0\}.
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Bertrand Competition - 3 firms

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Bertrand Competition - 3 firms

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Examples - Continued
  Cournot - Revisited
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  Hotelling and Voting Models
Hotelling

Two firms \( i = 1, 2 \) decide to produce heterogeneous products \( x_1, x_2 \in [0, 1] \).
Hotelling

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- $x_1, x_2$ represents the characteristic of the product
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- For example, this could be interpreted as a model in which there is a “linear city” represented by the interval $[0, 1]$. 
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In this interpretation, the firms are each deciding where to locate on this line
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- In this interpretation, the firms are each deciding where to locate on this line
- Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume
Hotelling

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Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume.

If the firms $i = 1, 2$ respectively produce products of characteristic $x_1$ and $x_2$, then a consumer at $\theta$ would consume whichever product is closest to $\theta$. 
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- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
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- Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume
- If the firms $i = 1, 2$ respectively produce products of characteristic $x_1$ and $x_2$, then a consumer at $\theta$ would consume whichever product is closest to $\theta$
- The game consists of the two players $i = 1, 2$, each of whom chooses a point $x_1, x_2 \in [0, 1]$ simultaneously.
Hotelling

\[ x_1, x - x_1, x_2 - x, 1 - x_2 \]

0 \( Firm 1 \) \( x \) \( Firm 2 \) 1
Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

\[
    u_1(x_1, x_2) = \begin{cases} 
    \frac{x_1 + x_2}{2} & \text{if } x_1 < x_2, \\
    \frac{1}{2} & \text{if } x_1 = x_2, \\
    1 - \frac{x_1 + x_2}{2} & \text{if } x_1 > x_2.
    \end{cases}
\]

Similarly,

\[
    u_2(x_1, x_2) = \begin{cases} 
    1 - \frac{x_1 + x_2}{2} & \text{if } x_1 < x_2, \\
    \frac{1}{2} & \text{if } x_1 = x_2, \\
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\end{cases}
\]
Compute the best response functions

- **Case 1:** Suppose first that \( x_2 > 1/2 \). Then setting \( x_1 \) against \( x_2 \) yields a payoff of

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\end{cases}
\]

This utility function has a discontinuity at \( x_1 = x_2 \) and jumps down to \( 1/2 \) at \( x_1 = x_2 \). There will be no best response for firm 1 (try to set as close to the left the other firm as possible).
Compute the best response functions

**Case 1:** Suppose first that $x_2 > 1/2$. Then setting $x_1$ against $x_2$ yields a payoff of

$$u_1(x_1, x_2) = \begin{cases} 
\frac{x_1 + x_2}{2} & \text{if } x_1 < x_2, \\
\frac{1}{2} & \text{if } x_1 = x_2, \\
1 - \frac{x_1 + x_2}{2} & \text{if } x_1 > x_2. 
\end{cases}$$

This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible).

**Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible).
Hotelling

Compute the best response functions

- **Case 1:** Suppose first that $x_2 > 1/2$. Then setting $x_1$ against $x_2$ yields a payoff of

  \[ u_1(x_1, x_2) = \begin{cases} 
  \frac{x_1 + x_2}{2} & \text{if } x_1 < x_2, \\
  \frac{1}{2} & \text{if } x_1 = x_2, \\
  1 - \frac{x_1 + x_2}{2} & \text{if } x_1 > x_2.
  \end{cases} \]

  This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible)

- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)

- **Case 3:** Suppose next that $x_2 = 1/2$. Here there will be a best response for firm 1 at $1/2$
Hotelling

\[ BR_1(x_2) = \begin{cases} 
\emptyset & \text{if } x_2 > 1/2 \\
1/2 & \text{if } x_2 = 1/2 \\
\emptyset & \text{if } x_2 < 1/2.
\end{cases} \]

Symmetrically, we have:

\[ BR_2(x_1) = \begin{cases} 
\emptyset & \text{if } x_1 > 1/2 \\
1/2 & \text{if } x_1 = 1/2 \\
\emptyset & \text{if } x_1 < 1/2.
\end{cases} \]

The unique Nash equilibrium is for each firm to choose \((x_1, x_2) = (1/2, 1/2)\). Each firm essentially locates in the same place.
Hotelling

- Hotelling can also be done in a discreet setting
- Hotelling can be applied to a variety of situations (e.g., voting)
- But this predicts the opposite of polarization
- With three candidates, predictions are quite different
- All candidates picking $\frac{1}{2}$ is no longer a Nash equilibrium
- What are the set of pure strategy equilibria here? (this is a difficult problem)