Lecture 12: Game Theory // Nash equilibrium

Mauricio Romero

Beauty contest

- Consider the following game among 100 people. Each individual selects a number, s_i, between 20 and 60.
- Let a_{-i} be the average of the number selected by the other 99 people. i.e. $a_{-i} = \sum_{j\ neqi} \frac{s_j}{99}$.
- The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}a_{-i})^2$

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$$-2(s_i - \frac{3}{2}a_{-i}) = 0$$

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- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- but $a_{-i} \in [20, 60]$
- Therefore $s_i = 20$ is dominated by $s_i = 30$

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- 60 would dominate any other selection and therefore all the players select 60.
- The solution by means of iterated elimination of dominated strategies is $\underbrace{(60, 60, ..., 60)}_{\text{100 times}}$

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

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- It's tempting to think player 1 would never play C
- However, if player 1 is sure that player two is going to play a he would be completely indifferent between playing B or C

Definition

s_i weakly dominates s_i' if for all opponent pure strategy profiles, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(\_i', s_{-i})$$

and there is at least one opponent strategy profile $s_{-i} \in S_{-i}$ for which

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

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- Rationality is not enough
- Even so, it sounds "logical" to do so and has the potential to greatly simplify a game
- There is a problem, and that is that the order in which we eliminate the strategies matters

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).

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A	3, 4	4, 3
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- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).
- If on the other hand, we notice that A is also weakly dominated by C then we can eliminate it in the first round, and this would eliminate a in the second round and therefore B would be eliminated. This would result in (C, b).

Remember the definition of competitive equilibrium in a market economy.

Definition

A competitive equilibrium in a market economy is a vector of prices and baskets x_i such that: 1) x_i maximizes the utility of each individual given the price vector i.e.

$$x_i = \arg \max_{p \ cdot x_i \leq p \cdot w_i} u(x_i)$$

2) the markets empty.

$$\sum_i x_i = \sum_i w_i$$

- 1) means that given the prices, individuals have no incentive to demand a different amount

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- The idea is to extend this concept to strategic situations

Best response

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Given a strategy profile of opponents s_{-i}, we can define the best response of player i:

$$BR_i(s_{-i}) = \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}).$$

- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$

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- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$
- There could be multiple strategies in $BR_i(s_{-i})$ but all such strategies give the same utility to player i if the opponents are indeed playing according to s_{-i}

Nash equilibrium

Definition

Suppose that we have a game $(I = \{1, 2, \dots n\}, S_1, \dots, S_n, u_1, \dots, u_n)$. Then a strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a **pure strategy** Nash equilibrium if for every i and for every $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

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- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- once this equilibrium is reached, nobody has incentives to move from there
- This is a concept of stability, but there is no way to ensure, or predict, that the game will reach this equilibrium

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Beauty contest

The best response of an individual is given by

$$s_i(s_{-i})^* = \begin{cases} \frac{3}{2}s_{-i} & \text{si } s_{-i} \leq 40 \\ 60 & \text{si } s_{-i} > 40 \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)

Prisoner's dilemma

	C	NC
C	5,5	0,10
NC	10,0	2,2

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The best response functions are:

$$BR_i(s_{-i}) = \begin{cases} NC & \text{si } s_i = C \\ NC & \text{si } s_i = NC \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play NC, NC)

Prisoner's dilemma – A trick

Best response of 1 to 2 playing C

	C	NC
C	5,5	0,10
NC	<u>10</u>,0	2,2

Prisoner's dilemma – A trick

Best response of 1 to 2 playing NC

	C	NC
C	5,5	0,10
NC	$\underline{10}$,0	$\underline{2}$,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing C

	C	NC
C	5,5	0,<u>10</u>
NC	<u>10</u>,0	<u>2</u>,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing NC

	C	NC
C	5,5	$0,\underline{10}$
NC	$\underline{10},0$	$\underline{2},\underline{2}$

When underlined for both players, it is a Nash equilibrium (both are doing their BR)

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

Battle of the sexes

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	G	P
G	$\underline{2},\underline{1}$	0,0
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$$BR_i(s_{-i}) = \begin{cases} G & \text{si } s_i = G \\ P & \text{si } s_i = P \end{cases}$$

Thus, (G, G) y (P, P) are both Nash equilibrium

Matching pennies (Pares o Nones I)

	1	2
1	(1000,-1000)	(-1000,1000)
2	(-1000,1000)	(1000,-1000)

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$$BR_1(s_2) = \begin{cases} 1 & \text{si } s_2 = 1 \\ 2 & \text{si } s_2 = 2 \end{cases}$$

$$BR_2(s_1) = \begin{cases} 2 & \text{si } s_1 = 1 \\ 1 & \text{si } s_2 = 2 \end{cases}$$

There is no Nash equilibrium in pure strategies

Lecture 12: Game Theory // Nash equilibrium

Nash equilibrium survive IDSDS

Theorem

Every Nash equilibrium survives the iterative elimination of strictly dominated strategies

Proof

By contradiction:

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$$u_i(s_i^*, s_{-i}) < u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

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- And this is a contradiction!

Nash equilibrium survive IDSDS

Theorem

If the process of IDSDS comes to a single solution, that solution is a Nash Equilibrium and is unique.

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

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- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium

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By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

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$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

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- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

- Strategy space is $S_i = [0, +\infty)$
- The utility function of player i is given by:

$$\pi_1(q_1, q_2) = (120 - (q_1 + q_2))q_1,$$
$$\pi_2(q_1, q_2) = (120 - (q_1 + q_2))q_2.$$

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- Therefore 60 strictly dominates any $q_i \in (60, 120]$

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- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$

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- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$
- Such a q_i can never be strictly dominated

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- After one round of deletion of strictly dominated strategies, we are left with: $S_i = [0, 60]$

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- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [0, 60]$
- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$

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- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [0, 60]$
- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$
- After two rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 60]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 60]$
- 45 strictly dominates all strategies $q_i \in (45, 60]$
- After three rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 45]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

- $q_{-i} = [30, 45]$

- 37.5 strictly dominates all strategies $q_i \in [37.5, 45]$

- After four rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 45]$

Cournot Competition

- After (infinitely) many iterations, the only remaining strategies are $S_i = 40$

- The unique solution by IDSDS is $q_1^* = q_2^* = 40$.

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- $$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$

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- At any Nash equilibrium, we must have: $q_1^* \in BR_1(q_2^*)$ and $q_2^* \in BR_2(q_1^*)$.
- $$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$
- We can solve for q_1^* and q_2^* to obtain:
$$q_1^* = 40, q_2^* = 40, Q^* = 80, \Pi_1^* = \Pi_2^* = 1600.$$

Cournot Competition vs Monopoly (cartel)

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$$\max_{Q}(120 - Q)Q \Rightarrow Q^{*} = 60, P^{*} = 60, \Pi^{m} = 3600.$$

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- The profits to each firm in the Cournot Competition is less than half of the monopoly profits

Cournot Competition vs Monopoly (cartel)

- In a perfectly competitive market, price equals marginal cost and the total quantity produced will be $Q = 120$.
- A monopolist would solve the following maximization problem:

$$\max_{Q}(120 - Q)Q \Rightarrow Q^* = 60, P^* = 60, \Pi^m = 3600.$$

- The profits to each firm in the Cournot Competition is less than half of the monopoly profits
- In a duopoly, externalities are imposed on the other firm

Cournot Competition - General case

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- $$\max_{q_i} p(q_i + Q_{-i})q_i - c_i(q_i)$$

Cournot Competition - General case

- First order condition implies:

$$
\begin{aligned}
q_i \frac{dP}{dQ}(q_i + Q_{-i}) + P(q_i + Q_{-i}) &= \frac{dc_i}{dq_i}(q_i) \\
\frac{P(Q) - \frac{dc_i}{dq_i}(q_i)}{P(Q)} &= -\frac{q_i}{Q}\frac{Q}{P(Q)}\frac{dP}{dQ}(Q)
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- Therefore in a pure strategy Nash equilibrium $(q_1^*, q_2^*, \dots, q_n^*)$ with $Q^* = q_1^* + q_2^* + \cdots q_n^*$, we must have:

$$\frac{P(Q^*) - \frac{dc_1}{dq_1}(q_1^*)}{P(Q^*)} = -\frac{q_1^*}{Q^*}\frac{1}{\varepsilon_{Q,P}(Q^*)},$$

$$\frac{P(Q^*) - \frac{dc_2}{dq_2}(q_2^*)}{P(Q^*)} = -\frac{q_2^*}{Q^*}\frac{1}{\varepsilon_{Q,P}(Q^*)},$$

$$\vdots$$

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Cournot Competition - General case

- Suppose that all firms have exactly the same cost function c

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- Rewriting

$$P(Q^*) = \frac{1}{1 + \frac{1}{n}\frac{1}{\varepsilon_{Q,P}(Q^*)}}\frac{\partial c}{dq}\left(\frac{Q^*}{n}\right).$$

Cartels

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- In a Nash equilibrium we must have:

$$\begin{aligned} q_1^* &= \frac{1 - q_2^* - q_3^*}{2} \\ q_2^* &= \frac{1 - q_1^* - q_3^*}{2} \\ q_3^* &= \frac{1 - q_1^* - q_2^*}{2}. \end{aligned}$$

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- $q_1^* = q_2^* = q_3^* = \frac{1}{4}$
- Price is $p^* = 1/4$ and all firms get the same profits of $1/16$

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- Therefore

$$q_A^* = \frac{1 - q_B^*}{2}$$
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- Firm 3 is obtaining a disproportionate share of the joint profits (more than $1/3$)

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- Firm 3 clearly wants to stay out

Cartels

- This exercise explains the difficulties associated with sustaining collusive agreements (e.g., the OPEC cartel)