Lecture 12: Game Theory // Nash equilibrium

Mauricio Romero

Lecture 12: Game Theory // Nash equilibrium

Examples - Continued

Mixed strategies

Examples - Continued

Mixed strategies

Bertrand Competition

- N identical firms competing on the same market

Bertrand Competition

- N identical firms competing on the same market
- Marginal cost is constant and equal to c

Bertrand Competition

- N identical firms competing on the same market
- Marginal cost is constant and equal to c
- Aggregate inverse demand is

$$p = a - b \sum_{j=1}^{N} q^j$$

Bertrand Competition

- N identical firms competing on the same market
- Marginal cost is constant and equal to c
- Aggregate inverse demand is

$$p = a - b \sum_{j=1}^{N} q^j$$

- Benefits of firm j are:

$$\Pi^j(q^1, ...q^N) = \left(a - b \sum_{i=1}^{N} q^i \right) q^j - cq^j.$$

Bertrand Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

Bertrand Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

- The symmetric Nash equilibrium is given by

$$q^* = \frac{a-c}{b(N+1)}$$

Bertrand Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

- The symmetric Nash equilibrium is given by

$$q^* = \frac{a-c}{b(N+1)}$$

- Thus

$$\begin{aligned} \sum_{j=1}^{N} q^j &= \frac{N(a-c)}{b(N+1)} \\ p &= a - N\frac{a-c}{(N+1)} < a \\ \Pi^j &= \frac{(a-c)^2}{b(N+1)^2} \end{aligned}$$

Bertrand Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$

$$p = a - N\frac{a-c}{(N+1)} < a$$

$$\Pi^j = \frac{(a-c)^2}{b(N+1)^2}$$

- As $N \to \infty$ we get close to perfect competition

Bertrand Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$

$$p = a - N\frac{a-c}{(N+1)} < a$$

$$\Pi^j = \frac{(a-c)^2}{b(N+1)^2}$$

- As $N \to \infty$ we get close to perfect competition
- $N = 1$ we get the monopoly case

Bertrand Competition

- Consider the alternative model in which firms set prices
- In the monopolist's problem, there was not distinction between a quantity-setting model and a price setting
- In oligopolistic models, this distinction is very important

Bertrand Competition

- Consider two firms with the same marginal constant marginal cost of production and demand is completely inelastic
- Each firm simultaneously chooses a price $p_i \in [0, +\infty)$
- If p_1, p_2 are the chosen prices, then the utility functions of firm i is given by:

$$u_i(p_i, p_{-i}) = \begin{cases} p_{-i} - \varepsilon & \text{if } p_i > p_{-i}, \\ (p_i - c)\frac{Q(p_i)}{2} & \text{if } p_i = p_{-i}, \\ (p_i - c)Q(p_i) & \text{if } p_i < p_{-i}. \end{cases}$$

Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$R(p_i) = p_i Q(p_i) \quad (1)$$

$$MR(p_i) = Q(p_i) + p_i Q'(p_i) \quad (2)$$

$$= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). \quad (3)$$

Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$
\begin{aligned}
R(p_i) &= p_i Q(p_i) && (1) \\
MR(p_i) &= Q(p_i) + p_i Q'(p_i) && (2) \\
&= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). && (3)
\end{aligned}
$$

- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$R(p_i) = p_i Q(p_i) \quad (1)$$
$$MR(p_i) = Q(p_i) + p_i Q'(p_i) \quad (2)$$
$$= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). \quad (3)$$

- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

- Then

$$MR(p_i) - c > 0 \text{ if } p_i < p^m, MR(p_i) - c < 0 \text{ if } p_i > p^m.$$

Bertrand Competition

- The best response function is:

$$BR_i(p_{-i}) = \begin{cases} p^m & \text{if } p_{-i} > p^m, \\ p_{-i} - \varepsilon & \text{if } c < p_{-i} \leq p^m, \\ [c, +\infty) & \text{if } c = p_{-i} \\ (c, +\infty) & \text{if } c > p_{-i}. \end{cases}$$

- Where ε is the smallest monetary unit

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$
- $BR_1(p^m - \varepsilon) = p^m - 2\varepsilon$

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$
- $BR_1(p^m - \varepsilon) = p^m - 2\varepsilon$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$
- $BR_1(p_1^* - \varepsilon) = p_1^* - 2\varepsilon$

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$
- $BR_1(p_1^* - \varepsilon) = p_1^* - 2\varepsilon$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) > c$

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) > c$
- $BR_1(p_2^*) = p_2^* - \varepsilon$

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) > c$
- $BR_1(p_2^*) = p_2^* - \varepsilon$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$

Bertrand Competition

Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$
- The unique pure strategy Nash equilibrium is $p_1^* = p_2^* = c$

Bertrand Competition

Thus in contrast to the Cournot duopoly model, in the Bertrand competition model, two firms get us back to perfect competition ($p = c$)

Bertrand Competition - different costs

- Suppose that the marginal cost of firm 1 is equal to c_1 and the marginal cost of firm 2 is equal to c_2 where $c_1 < c_2$.
- The best response for each firm:

$$BR_i(p_{-i}) = \begin{cases} p_m^i & \text{if } p_{-i} > p_m^i, \\ p_{-i} - \varepsilon & \text{if } c_i < p_{-i} \leq p_m^i, \\ [c_i, +\infty) & \text{if } p_{-i} = c_i \\ (p_{-i}, +\infty) & \text{if } p_{-i} < c_i. \end{cases}$$

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 1 would be making a loss

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 1 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 1 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 1 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit
- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 1 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit
- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices
- No NE because of continuous prices

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.
- The demand function is given by:

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.
- The demand function is given by:
- Suppose that (p_1^*, p_2^*) is a pure strategy Nash equilibrium...

Bertrand Competition - discreet prices

Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$

Bertrand Competition - discreet prices

Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$
- p_1^* cannot be a best response to p_2^* since by setting $p_1 = p_2^*$ firm 1 would get strictly positive profits

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price
- The only equilibrium is $(p_1^*, p_1^* + 1)$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher
- $(p_1^*, p_2^*) = (10, 11)$ is a Nash equilibrium

Bertrand Competition - discreet prices

Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$

Bertrand Competition - discreet prices

Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$
- Firm 1 would not be best responding since by setting a price of $p_1 = 10$, it would get strictly positive profits

Bertrand Competition - discreet prices

Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$

Bertrand Competition - discreet prices

Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$
- Firm 1 is not best responding since by lowering the price it can get the whole market.

Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms

Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms
- Best response of firm i is given by:

$$BR_1(p_2, p_3) = \begin{cases} p^m & \text{if } \min\{p_2, p_3\} > p^m, \\ \min\{p_2, p_3\} - \varepsilon & \text{if } c < \min\{p_2, p_3\} \leq p^m, \\ [c, +\infty) & \text{if } c = \min\{p_2, p_3\}, \\ (\min\{p_2, p_3\}, +\infty) & \text{if } c > \min\{p_2, p_3\}. \end{cases}$$

Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms
- Best response of firm i is given by:

$$BR_1(p_2, p_3) = \begin{cases} p^m & \text{if } \min\{p_2, p_3\} > p^m, \\ \min\{p_2, p_3\} - \varepsilon & \text{if } c < \min\{p_2, p_3\} \leq p^m, \\ [c, +\infty) & \text{if } c = \min\{p_2, p_3\}, \\ (\min\{p_2, p_3\}, +\infty) & \text{if } c > \min\{p_2, p_3\}. \end{cases}$$

- (c, c, c) is indeed a pure strategy Nash equilibrium as in the two firm case

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c?

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c?

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost
- Set of all pure strategy Nash equilibria are given by:

$$\{(c, c, c+\varepsilon) : \varepsilon \geq 0\} \cup \{(c, c+\varepsilon, c) : \varepsilon \geq 0\} \cup \{(c+\varepsilon, c, c) : \varepsilon \geq 0\}.$$

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product
- For example, this could be interpreted as a model in which there is a "linear city" represented by the interval $[0, 1]$

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product
- For example, this could be interpreted as a model in which there is a "linear city" represented by the interval $[0, 1]$
- In this interpretation, the firms are each deciding where to locate on this line

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product
- For example, this could be interpreted as a model in which there is a "linear city" represented by the interval $[0, 1]$
- In this interpretation, the firms are each deciding where to locate on this line
- Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume

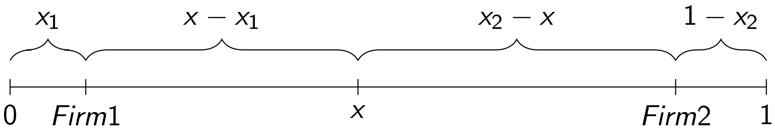
Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product
- For example, this could be interpreted as a model in which there is a "linear city" represented by the interval $[0, 1]$
- In this interpretation, the firms are each deciding where to locate on this line
- Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume
- If the firms $i = 1, 2$ respectively produce products of characteristic x_1 and x_2, then a consumer at θ would consume whichever product is closest to θ

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$
- x_1, x_2 represents the characteristic of the product
- For example, this could be interpreted as a model in which there is a "linear city" represented by the interval $[0, 1]$
- In this interpretation, the firms are each deciding where to locate on this line
- Consumers are uniformly distributed on the line $[0, 1]$, where $\theta \in [0, 1]$ represents the consumers ideal type of product that he would like to consume
- If the firms $i = 1, 2$ respectively produce products of characteristic x_1 and x_2, then a consumer at θ would consume whichever product is closest to θ
- The game consists of the two players $i = 1, 2$, each of whom chooses a point $x_1, x_2 \in [0, 1]$ simultaneously.

Hotelling



Hotelling

Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Similarly,

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Hotelling

Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Similarly,

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Hotelling

Compute the best response functions

- **Case 1:** Suppose first that $x_2 > 1/2$. Then setting x_1 against x_2 yields a payoff of

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible)

Hotelling

Compute the best response functions

- **Case 1:** Suppose first that $x_2 > 1/2$. Then setting x_1 against x_2 yields a payoff of

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

  This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible)

- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)

Hotelling

Compute the best response functions

- **Case 1:** Suppose first that $x_2 > 1/2$. Then setting x_1 against x_2 yields a payoff of

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

  This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible)

- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)

- **Case 3:** Suppose next that $x_2 = 1/2$. Here there will be a best response for firm 1 at $1/2$

Hotelling

$$BR_1(x_2) = \begin{cases} \emptyset & \text{if } x_2 > 1/2 \\ 1/2 & \text{if } x_2 = 1/2 \\ \emptyset & \text{if } x_2 < 1/2. \end{cases}$$

Symmetrically, we have:

$$BR_2(x_1) = \begin{cases} \emptyset & \text{if } x_1 > 1/2 \\ 1/2 & \text{if } x_1 = 1/2 \\ \emptyset & \text{if } x_1 < 1/2. \end{cases}$$

The unique Nash equilibrium is for each firm to choose $(x_1, x_2) = (1/2, 1/2)$. Each firm essentially locates in the same place

Hotelling

- Hotelling can also be done in a discreet setting
- Hotelling can be applied to a variety of situations (e.g., voting)
- But this predicts the opposite of polarization
- With three candidates, predictions are quite different
- All candidates picking $\frac{1}{2}$ is no longer a Nash equilibrium
- What are the set of pure strategy equilibria here? (this is a difficult problem).

Lecture 12: Game Theory // Nash equilibrium

Examples - Continued

Mixed strategies

Examples - Continued

Mixed strategies

Mixed strategies

Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

- This game is entirely stochastic (ability has nothing to do with your chances of winning)

Mixed strategies

Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

- This game is entirely stochastic (ability has nothing to do with your chances of winning)
- The probability of winning with every strategy is the same

Mixed strategies

Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

- This game is entirely stochastic (ability has nothing to do with your chances of winning)
- The probability of winning with every strategy is the same
- Thus, people *tend* choose randomly which of the three options to play

Mixed strategies

Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

- This game is entirely stochastic (ability has nothing to do with your chances of winning)
- The probability of winning with every strategy is the same
- Thus, people *tend* choose randomly which of the three options to play
- We would like the concept of Nash equilibrium to reflect this

Mixed strategies

Definition

A mixed strategy σ_i is a function $\sigma_i : S_i \to [0,1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

- $\sigma_i(s_i)$ represents the probability with which player i plays s_i

Mixed strategies

Definition

A mixed strategy σ_i is a function $\sigma_i : S_i \to [0, 1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

- $\sigma_i(s_i)$ represents the probability with which player i plays s_i
- A **pure strategy** is simply a mixed strategy σ_i that plays some strategy $a_i \in S_i$ with probability one

Mixed strategies

Definition

A mixed strategy σ_i is a function $\sigma_i : S_i \to [0, 1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

- $\sigma_i(s_i)$ represents the probability with which player i plays s_i
- A **pure strategy** is simply a mixed strategy σ_i that plays some strategy $a_i \in S_i$ with probability one
- We will denote the set of all mixed strategies of player i by Σ_i

Mixed strategies

- Given a mixed strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we need a way to define how players evaluate payoffs of mixed strategy profiles

Mixed strategies

- Given a mixed strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we need a way to define how players evaluate payoffs of mixed strategy profiles
- $$u_1(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s \in S} u_1(s_1, s_2, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n).$$

Mixed strategies

- Given a mixed strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we need a way to define how players evaluate payoffs of mixed strategy profiles
- $$u_1(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s \in S} u_1(s_1, s_2, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2) \cdots \sigma_n(s_n).$$
- For instance, assume my opponent is playing randomizing over paper and scissors with probability $\frac{1}{2}$ (i.e., $\sigma_{-i} = (0, \frac{1}{2}, \frac{1}{2})$)

Mixed strategies

- Given a mixed strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we need a way to define how players evaluate payoffs of mixed strategy profiles
- $$u_1(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s \in S} u_1(s_1, s_2, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n).$$
- For instance, assume my opponent is playing randomizing over paper and scissors with probability $\frac{1}{2}$ (i.e., $\sigma_{-i} = (0, \frac{1}{2}, \frac{1}{2})$)
- The expected utility of playing "rock" is

$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$

Mixed strategies

- Given a mixed strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we need a way to define how players evaluate payoffs of mixed strategy profiles
- $$u_1(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s \in S} u_1(s_1, s_2, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n).$$
- For instance, assume my opponent is playing randomizing over paper and scissors with probability $\frac{1}{2}$ (i.e., $\sigma_{-i} = (0, \frac{1}{2}, \frac{1}{2})$)
- The expected utility of playing "rock" is

$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$

- If I'm randomizing over rock and scissors (i.e., $s_i = (\frac{1}{2}, 0, \frac{1}{2})$) then

$$E(U_i(\sigma, \sigma_{-i})) = \underbrace{-1\frac{1}{4}}_{\text{rock vs paper}} + \underbrace{-1\frac{1}{4}}_{\text{rock vs scissors}} + \underbrace{1\frac{1}{4}}_{\text{scissors vs paper}} + \underbrace{0\frac{1}{4}}_{\text{scissors vs scissors}} = -\frac{1}{4}$$

Mixed strategies

Definition

A (possibly mixed) strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n)^*$ is a Nash equilibrium if and only if for every i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Sigma_i$.

Mixed strategies

Definition (Mixed Strategy Dominance Definition A)

Let σ_i, σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if for all mixed strategies of the opponents, σ_{-i},

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}).$$

Mixed strategies

If σ_i is better than σ_i' no matter what **pure strategy** opponents play, then σ_i is also strictly better than σ_i' no matter what **mixed strategies** opponents play

Theorem

Let σ_i and σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if and only if for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

Proof- Part 1

- Since $S_{-i} \subseteq \Sigma_{-i}$, if σ_i strictly dominates σ_i'

Proof- Part 1

- Since $S_{-i} \subseteq \Sigma_{-i}$, if σ_i strictly dominates σ_i'

- Then for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

Proof - Part 2

- To prove the other direction, suppose that for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

Proof - Part 2

- To prove the other direction, suppose that for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma'_i, s_{-i}).$$

- For any σ_{-i},

$$\begin{aligned}
u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, s_{-i}) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i})
\end{aligned}$$

Proof - Part 2

- To prove the other direction, suppose that for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

- For any σ_{-i},

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) \end{aligned}$$

- So

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) > \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i', s_{-i}) = u_i(\sigma_i', \sigma_{-i})$$

Definition (Mixed Strategy Dominance Definition B)

Let σ_i, σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if for all pure strategies of the opponents, $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

Battle of the sexes