Lecture 13: Game Theory // Nash equilibrium

Mauricio Romero

Lecture 13: Game Theory // Nash equilibrium

Mixed strategies

Examples

Nash's Theorem

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Nash's Theorem

Mixed strategies

Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

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- Thus, people *tend* choose randomly which of the three options to play
- We would like the concept of Nash equilibrium to reflect this

Mixed strategies

Definition

A mixed strategy σ_i is a function $\sigma_i : S_i \to [0,1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

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- We will denote the set of all mixed strategies of player i by Σ_i

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- The expected utility of playing "rock" is
$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$

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- For instance, assume my opponent is playing randomizing over paper and scissors with probability $\frac{1}{2}$ (i.e., $\sigma_{-i} = (0, \frac{1}{2}, \frac{1}{2})$)
- The expected utility of playing "rock" is

$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$

- If I'm randomizing over rock and scissors (i.e., $s_i = (\frac{1}{2}, 0, \frac{1}{2})$) then

$$E(U_i(\sigma, \sigma_{-i})) = \underbrace{-1\frac{1}{4}}_{\text{rock vs paper}} + \underbrace{-1\frac{1}{4}}_{\text{rock vs scissors}} + \underbrace{1\frac{1}{4}}_{\text{scissors vs paper}} + \underbrace{0\frac{1}{4}}_{\text{scissors vs scissors}} = -\frac{1}{4}$$

Mixed strategies

Definition

A (possibly mixed) strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n)^*$ is a Nash equilibrium if and only if for every i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Sigma_i$.

Mixed strategies

Definition (Mixed Strategy Dominance Definition A)

Let σ_i, σ'_i be two mixed strategies of player i. Then σ_i strictly dominates σ'_i if for all mixed strategies of the opponents, σ_{-i},

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i}).$$

Mixed strategies

If σ_i is better than σ_i' no matter what **pure strategy** opponents play, then σ_i is also strictly better than σ_i' no matter what **mixed strategies** opponents play

Theorem

Let σ_i and σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if and only if for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

Proof- Part 1

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- Then for all $s_{-i} \in S_{-i}$,

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Proof - Part 2

- To prove the other direction, suppose that for all $s_{-i} \in S_{-i}$,

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- For any σ_{-i},

$$\begin{aligned}
u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_{-i}) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(\sigma_i, s_{-i})
\end{aligned}$$

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- For any σ_{-i},

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- So

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) > \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i', s_{-i}) = u_i(\sigma_i', \sigma_{-i})$$

Mixed strategies

Definition (Mixed Strategy Dominance Definition B)

Let σ_i, σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if for all pure strategies of the opponents, $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

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G	2,1	0,0
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- There are two pure strategy equilibria (G, G) and (P, P)
- We now look for Nash equilibria that involve randomizationby the players

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- **Case 3:** If $q < 1/3$, then $2q < 2/3 < 1 - q$ and therefore the best response is $p = 0$
- Thus, the best response function is given by:

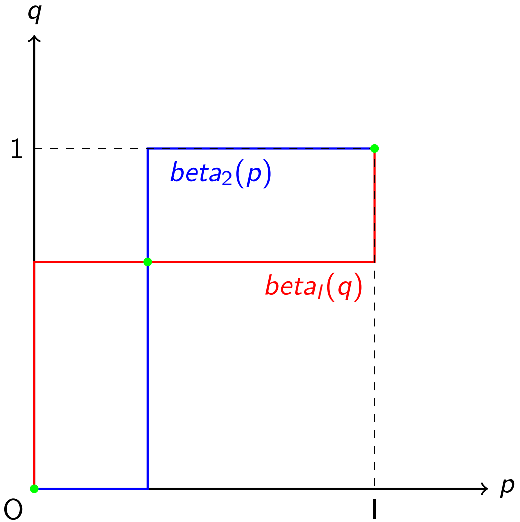
$$BR_1(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0, 1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3. \end{cases}$$

Battle of the sexes

Similarly we can calculate the best response function for player 2 and we get:

$$BR_2(p) = \begin{cases} 1 & \text{if } p > 2/3 \\ [0,1] & \text{if } p = 2/3 \\ 0 & \text{if } p < 2/3. \end{cases}$$

Battle of the sexes



- Thus, there are three points where the best response curves cross: $(1, 1), (0, 0,), (2/3, 1/3)$

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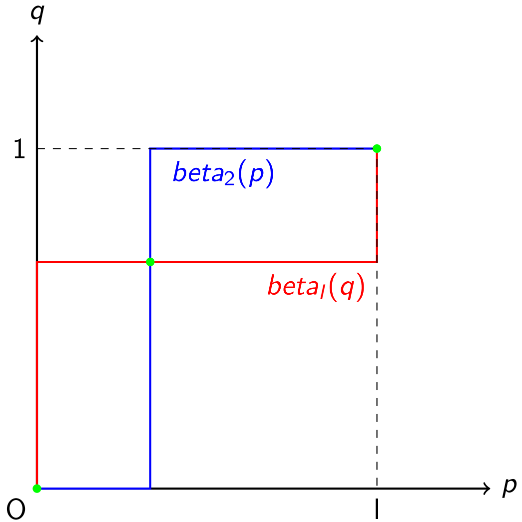

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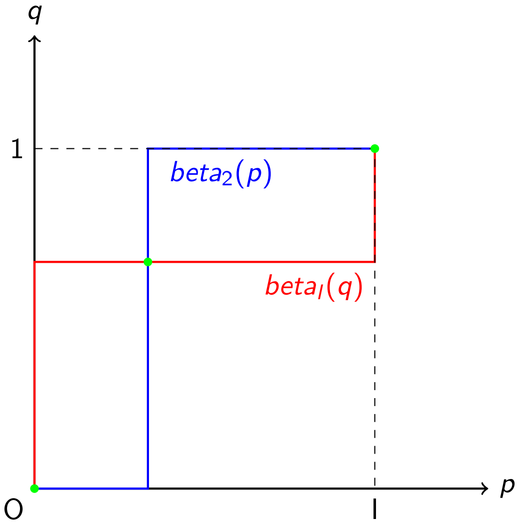

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Consider the following game

	E	F	G
A	5, 10	5, 3	3, 4
B	1, 4	7, 2	7, 6
C	4, 2	8, 4	3, 8
D	2, 4	1, 3	8, 4

- Consider $\sigma_1 = (\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}))$

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- Then $BR_2(\theta_1) = \{(p, 0, 1-p), p \in [0, 1]\}$

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- D dominates B (player 1)

Reduced game

	E	G
A	5, 10	3, 4
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- Note that $\sigma_1 = (p, 0, 1-p)$ with $p > \frac{2}{3}$ dominates C
- $\mathbb{E}U(\sigma_1, E) = 5p + 2(1-p) = 3p + 2$
- $\mathbb{E}U(\sigma_1, G) = 3p + 8(1-p) = 8 - 3p$
-

$$\begin{aligned} \mathbb{E}U(\sigma_1, E) &> U(C, E) \\ 3p + 2 &> 4 \\ p &> \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbb{E}U(\sigma_1, G) &> \mathbb{E}U(C, G) \\ 8 - 3p &> 3 \\ p &< \frac{5}{3} \end{aligned}$$

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- $8 - 6q < 2q + 3$ if $\frac{5}{8} < q$
- Thus

$$BR_1(q, 1-q) = \begin{cases} \sigma_1 = (0,1) & \text{if } 0 \leq q < \frac{5}{8} \\ \sigma_1 = (1,0) & \text{if } \frac{5}{8} > q \geq 1 \\ \sigma_1 = (p, 1-p) & \text{if } \frac{5}{8} = q \end{cases}$$

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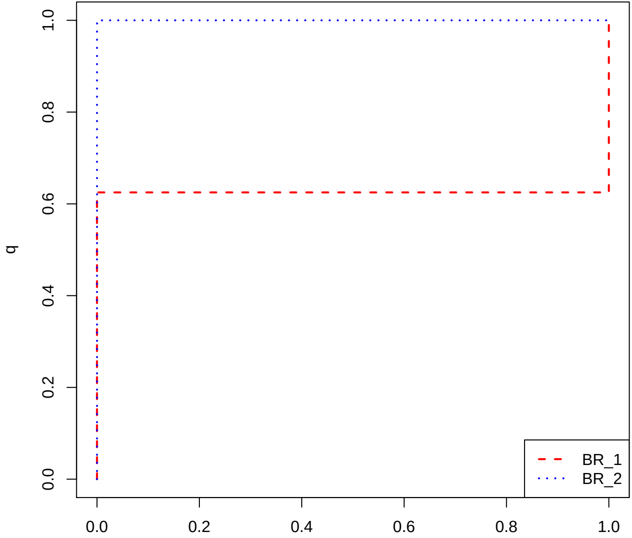
$$BR_2(p, 1-p) = \begin{cases} \sigma_2 = (1,0) & \text{if } p > 0 \\ \sigma_2 = (q, 1-q) & \text{if } p = 0 \end{cases}$$

Best responses



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Theorem (Nash's Theorem)

Suppose that the pure strategy set S_i is finite for all players i. A Nash equilibrium always exists.

Proof (just the intuition)

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Proof (just the intuition)

- Proof is very similar to general equilibrium proof
- Two parts:
  1. A Nash equilibrium is a fixed point of the best response functions
  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point
- Remember X^* is a fixed point of $F(X)$ if and only if $F(X^*) = X^*$

Proof - Part 1

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- Let $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$

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- Let $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$
- $\Gamma(s_1^*, ..., s_n^*) = (s_1^*, ..., s_n^*)$
- Therefore $(s_1^*, ..., s_n^*)$ is a fixed point of Γ

Proof - Part 2

Theorem (Kakutani fixed-point theorem)

Let $\Gamma : \Omega \to \Omega$ *be a correspondence that is upper semi-continuous,* Ω *be non empty, compact (closed and bounded), and convex* $\Rightarrow \Gamma$ *has at least one fixed point*

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- $\Gamma : \Sigma \rightarrow \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
- $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$ is upper semi-continous. Why?

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \to \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
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  - If two pure strategies are in the best response of a player $(s_i, s_i' \in BR_i(s_{-i}))$, then any mixing of those strategies is also a best response (i.e., $p\sigma + (1-p)\sigma \in BR_i(s_{-i})$)

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  - Therefore if $\Gamma(s_1, ..., s_n)$ has two images, those two images are connected (via all the mixed strategies that connect those two images)
- That happens to be the definition of upper semi-continous