Lecture 13: Game Theory // Nash equilibrium

Mauricio Romero

Applications of Subgame Perfect Nash Equilibrium

Applications of Subgame Perfect Nash Equilibrium

Theorem (Nash's Theorem)

Suppose that the pure strategy set S_i is finite for all players i. A Nash equilibrium always exists.

Proof (just the intuition)

- Proof is very similar to general equilibrium proof

Proof (just the intuition)

- Proof is very similar to general equilibrium proof
- Two parts:

Proof (just the intuition)

- Proof is very similar to general equilibrium proof
- Two parts:
  1. A Nash equilibrium is a fixed point of the best response functions

Proof (just the intuition)

- Proof is very similar to general equilibrium proof
- Two parts:
  1. A Nash equilibrium is a fixed point of the best response functions
  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point

Proof (just the intuition)

- Proof is very similar to general equilibrium proof
- Two parts:
  1. A Nash equilibrium is a fixed point of the best response functions
  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point
- Remember X^* is a fixed point of $F(X)$ if and only if $F(X^*) = X^*$

Proof - Part 1

- Let $(s_1^*, ..., s_n^*)$ be a Nash equilibrium

Proof - Part 1

- Let $(s_1^*, ..., s_n^*)$ be a Nash equilibrium
- Then $s_i^* = BR_i(s_{-i}^*)$ for all i

Proof - Part 1

- Let $(s_1^*, ..., s_n^*)$ be a Nash equilibrium
- Then $s_i^* = BR_i(s_{-i}^*)$ for all i
- Let $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$

Proof - Part 1

- Let $(s_1^*, ..., s_n^*)$ be a Nash equilibrium
- Then $s_i^* = BR_i(s_{-i}^*)$ for all i
- Let $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$
- $\Gamma(s_1^*, ..., s_n^*) = (s_1^*, ..., s_n^*)$

Proof - Part 1

- Let $(s_1^*, ..., s_n^*)$ be a Nash equilibrium
- Then $s_i^* = BR_i(s_{-i}^*)$ for all i
- Let $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$
- $\Gamma(s_1^*, ..., s_n^*) = (s_1^*, ..., s_n^*)$
- Therefore $(s_1^*, ..., s_n^*)$ is a fixed point of Γ

Proof - Part 2

Theorem (Kakutani fixed-point theorem)

Let $\Gamma : \Omega \rightarrow \Omega$ be a correspondence that is upper semi-continuous, Ω be non empty, compact (closed and bounded), and convex $\Rightarrow \Gamma$ has at least one fixed point

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \to \Sigma$

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \to \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \rightarrow \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \to \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
- $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$ is upper semi-continous. Why?

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \rightarrow \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
- $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$ is upper semi-continous. Why?
  - If two pure strategies are in the best response of a player ($s_i, s_i' \in BR_i(s_{-i})$), then any mixing of those strategies is also a best response (i.e., $p\sigma + (1-p)\sigma \in BR_i(s_{-i})$)

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \to \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
- $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$ is upper semi-continous. Why?
  - If two pure strategies are in the best response of a player ($s_i, s_i' \in BR_i(s_{-i})$), then any mixing of those strategies is also a best response (i.e., $p\sigma + (1-p)\sigma \in BR_i(s_{-i})$)
  - Therefore if $\Gamma(s_1, ..., s_n)$ has two images, those two images are connected (via all the mixed strategies that connect those two images)

Proof - Part 2

So we want to apply Kakutani's theorem. If the game is finite and we allow mixed strategies then

- $\Gamma : \Sigma \rightarrow \Sigma$
- Σ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
- Σ is convex: By allowing mixed strategies, we automatically make it convex
- $\Gamma(s_1, ..., s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), ..., BR_n(s_{-n}))$ is upper semi-continous. Why?
    - If two pure strategies are in the best response of a player ($s_i, s_i' \in BR_i(s_{-i})$), then any mixing of those strategies is also a best response (i.e., $p\sigma + (1-p)\sigma \in BR_i(s_{-i})$)
    - Therefore if $\Gamma(s_1, ..., s_n)$ has two images, those two images are connected (via all the mixed strategies that connect those two images)
- That happens to be the definition of upper semi-continous

Applications of Subgame Perfect Nash Equilibrium

Applications of Subgame Perfect Nash Equilibrium

- Dynamic game are those that capture a dynamic element in which some players know what others did before playing

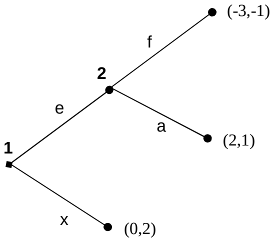
- Dynamic game are those that capture a dynamic element in which some players know what others did before playing
- Reminder: A (pure) strategy is a **complete contingent plan** of action at every information set

- Dynamic game are those that capture a dynamic element in which some players know what others did before playing

- Reminder: A (pure) strategy is a **complete contingent plan** of action at every information set

- The set of Nash equilibria of the extensive form game is simply the set of all Nash equilibria of the normal form representation of the game

- Dynamic game are those that capture a dynamic element in which some players know what others did before playing
- Reminder: A (pure) strategy is a **complete contingent plan** of action at every information set
- The set of Nash equilibria of the extensive form game is simply the set of all Nash equilibria of the normal form representation of the game
- Some of the equilibria do not make much sense intuitively


(-3,-1)
f
2
e
a
(2,1)
1
x
(0,2)

	f	a
e	-3,-1	2,1
x	0,2	0,2

	f	a
e	-3,-1	2,1
x	0,2	0,2

Two Nash equilibria: (x,f) y (e,a).

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But f is not a credible strategy

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But f is not a credible strategy
- If Firm 1 enters the market, Firm 2 will accommodate

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But f is not a credible strategy
- If Firm 1 enters the market, Firm 2 will accommodate
- We will study a refinement that will get rid of these type of equilibria

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But f is not a credible strategy
- If Firm 1 enters the market, Firm 2 will accommodate
- We will study a refinement that will get rid of these type of equilibria
- The overall idea is that agents must play an optimal action in each node

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But *f* is not a credible strategy
- If Firm 1 enters the market, Firm 2 will accommodate
- We will study a refinement that will get rid of these type of equilibria
- The overall idea is that agents must play an optimal action in each node
- In other words, play an optimal action in each node, conditional on reaching such node

- But (x,f) is a Nash equilibrium only because Firm 2 threatens to do a price war
- But f is not a credible strategy
- If Firm 1 enters the market, Firm 2 will accommodate
- We will study a refinement that will get rid of these type of equilibria
- The overall idea is that agents must play an optimal action in each node
- In other words, play an optimal action in each node, conditional on reaching such node
- In the previous example, f is not optimal if we reach the second period

- A natural way to make sure players are optimizing in each node is to solve the game via backwards induction

- A natural way to make sure players are optimizing in each node is to solve the game via backwards induction

- This amounts to starting from the end of the game, and work the way backwards by eliminating non-optimal strategies

- A natural way to make sure players are optimizing in each node is to solve the game via backwards induction

- This amounts to starting from the end of the game, and work the way backwards by eliminating non-optimal strategies

- A natural way to make sure players are optimizing in each node is to solve the game via backwards induction

- This amounts to starting from the end of the game, and work the way backwards by eliminating non-optimal strategies

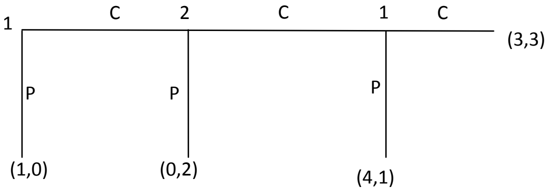
Theorem (Zermelo)

In every finite game where every information set has a single node (i.e., complete information), has an Nash equilibrium that can be derived via backwards induction. If the payouts to players are different in all terminal nodes, then the Nash equilibrium is unique.

Theorem (Zermelo II)

In any finite two-person game of perfect information in which the players move alternatingly and in which chance does not affect the decision making process, if the game cannot end in a draw, then one of the two players must have a winning strategy (i.e. force a win).

Centipede Game



	C	P
C,C	3,**3**	0,2
C,P	**4**,1	0,**2**
P,C	1,**0**	**1**,**0**
P,P	1,**0**	**1**,**0**

- Nash equilibria are $\{(P,P),P\}$ and $\{(P,C),P\}$

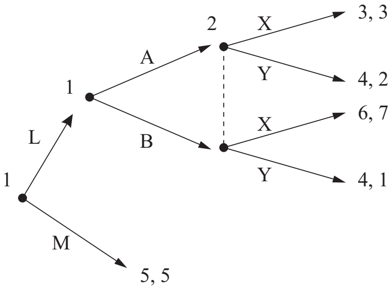
	C	P
C,C	3,**3**	0,2
C,P	**4**,1	0,**2**
P,C	1,**0**	**1**,**0**
P,P	1,**0**	**1**,**0**

- Nash equilibria are $\{(P, P), P\}$ and $\{(P, C), P\}$
- But if the game repeats 1,000 times it would be impossible to analyze

	C	P
C,C	3,**3**	0,2
C,P	**4**,1	0,**2**
P,C	1,**0**	**1**,**0**
P,P	1,**0**	**1**,**0**

- Nash equilibria are $\{(P, P), P\}$ and $\{(P, C), P\}$
- But if the game repeats 1,000 times it would be impossible to analyze
- But by backward induction, the solution is to play P in each period

Consider the following game



- Can't be solved by backwards induction

- Can't be solved by backwards induction

- Thus, we need something else

- Can't be solved by backwards induction

- Thus, we need something else

- First, we need to defined a subgame

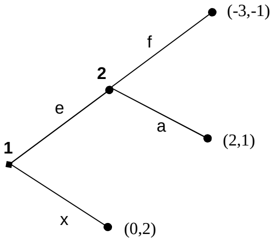
A sub-game, of a game in extensive form, is a sub-tree such that

- It starts in a single node
- If contains a node, it contains all subsequent nodes
- If it contains a node in an information set, it contains all nodes in the information set

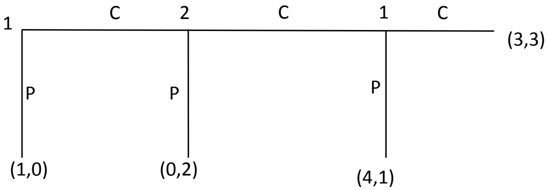
Definition

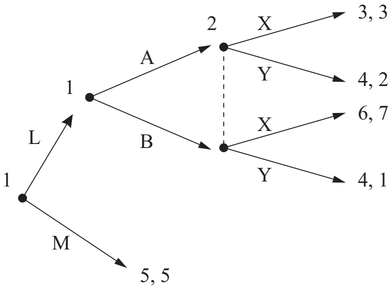
A subgame of an extensive form game is the set of all actions and nodes that follow a particular node that is not included in an information set with another distinct node

By definition, the original game is a subgame


(-3,-1)
f
2
e
a
(2,1)
1
x
(0,2)

Centipede Game




2
X
3, 3
A
Y
4, 2
1
6, 7
X
L
B
1
Y
4, 1
M
5, 5

Since in some games (where multiple nodes are in the same information set) we can't formally choose how people are optimizing, we extend the notion of backwards induction to subgames

Definition (Subgame perfect Nash equilibria)

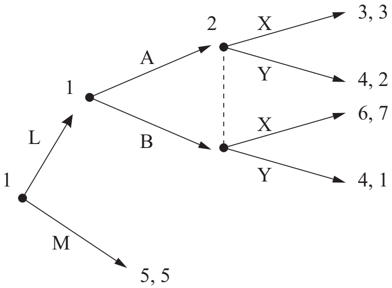
A pure strategy profile is a Subgame perfect Nash equilibria (SPNE) if and only if it involves the play of a NE in every subgame of the game.

Remark

Every SPNE is a NE

Remark

As in normal form games, mixed strategy SPNE can be defined but this is a bit technical. Thus, we will not worry about it for the purposes of the course.


1
L
M
5, 5
1
A
B
2
X
Y
X
Y
3, 3
4, 2
6, 7
4, 1

1 \ 2	X	Y
LA	3, 3	4, 2
LB	6, 7	4, 1
MA	5, 5	5, 5
MB	5, 5	5, 5

1 \ 2	X	Y
A	3, 3	4, 2
B	6, 7	4, 1

- The game has 3 NE: (LB,X), (MA,Y),(MB,Y)
- The subgame has a single NE: (B,X)
- The SPNE is (LB,X)