Lecture 17: Applications of Subgame Perfect Nash Equilibrium

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Lecture 1617 Applications of Subgame Perfect Nash Equilibrium

Entry Deterrence in Quantity Competition

Repeated Games

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  3. If potential entrant enters, she chooses $q_E \in [0, A]$.
  4. The price P is determined according to the inverse demand function $P = A - q_M - q_E$ and firms receive their payoffs, where there are no other costs to production than the one just mentioned above.

If the monopolist faced no entrant, then he would solve the following maximization problem:

$$\max_{q_M}(A - q_M)q_M \Longrightarrow q_M^m = \frac{A}{2}.$$

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- The entrant's best response function is:

$$BR_E(q_M) = \begin{cases} 0 & \text{if } \frac{(A-q_M)^2}{4} < F, \\ \{0, (A - q_M)/2\} & \text{if } \frac{(A-q_M)^2}{4} = F, \\ (A - q_M)/2 & \text{if } \frac{(A-q_M)^2}{4} > F. \end{cases}$$

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- This simplifies to

$$BR_E(q_M) = \begin{cases} 0 & \text{if } q_M > A - 2\sqrt{F}, \\ \{0, (A - q_M)/2\} & \text{if } q_M = A - 2\sqrt{F}, \\ (A - q_M)/2 & \text{if } q_M < A - 2\sqrt{F}. \end{cases}$$

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- Therefore by producing $q_M = A/2$, the entrant stays out
- The monopolist's best response is to choose $q_M^* = A/2$

Two SPNE:

$$q_M^* = A/2, q_E^*(q_M) = \begin{cases} 0 & \text{if } q_M > A - 2\sqrt{F}, \\ 0 & \text{if } q_M = A - 2\sqrt{F}, \\ (A - q_M)/2 & \text{if } q_M < A - 2\sqrt{F}. \end{cases}$$

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- Both lead to the same equilibrium outcome which is for the monopolist to choose $q_M = A/2$ and for the entrant to choose $q_E = 0$

- If the fixed cost is too high for entry, then even when the monopolist chooses the monopolist quantity, the entrant still wants to stay out

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- For simplicity we will look for equilibria in which the entrant chooses $(A - q_M)/2$ if $q_M = A - 2\sqrt{F}$

- As a result,

$$q_E^*(q_M) = \begin{cases} 0 & \text{if } q_M > A - 2\sqrt{F}, \\ \frac{A - q_M}{2} & \text{if } q_M \leq A - 2\sqrt{F}. \end{cases}$$

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- The monopolist must maximize the following utility function given that the entrant chooses according to q_E^*:

$$\max_{q_M \geq 0} u_M(q_M, q_E^*(\cdot)) = \max_{q_M \geq 0} \begin{cases} (A - q_M)q_M & \text{if } q_M > A - 2\sqrt{F} \\ \frac{q_M(A-q_M)}{2} & \text{if } q_M \leq A - 2\sqrt{F}. \end{cases}$$

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- The profits of the two firms are given by:

$$\pi_M = \frac{A^2}{8}, \pi_E = \frac{A^2}{16} - F.$$

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- We will use (G, T) to denote that game G is repeated T times

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4. After time T, if the action profiles chosen in times $1, 2, \ldots, T$ are given by $\big((a_i^1, a_{-i}^1), \ldots, (a_i^T, a_{-i}^T)\big)$:

$$\sum_{t=1}^{T} \delta^{t-1} u_i(a_i^t, a_{-i}^t).$$

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- Working incurs a cost of 1 however increases the utility of the other player $-i$ by 2
- Thus,

$$u_i(e_i, e_{-i}) = 2e_{-i} - e_i.$$

Prisoner's Dilemma (Game G)

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

- What happens when $T = 1$

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- NE: Players 1 and 2 will both choose $(e_1 = 0, e_1 = 0)$

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3. Then payoffs are realized as the discounted sum of the utilities of the actions in each period with discount factor $\delta \in (0, 1]$.

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$$u_1 = 1 + \delta \cdot 2$$
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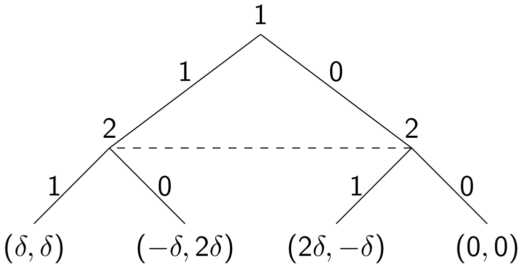
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 (2^5) pure strategies
- Similarly, player 2 has a total of 32 pure strategies

- There are 5 subgames



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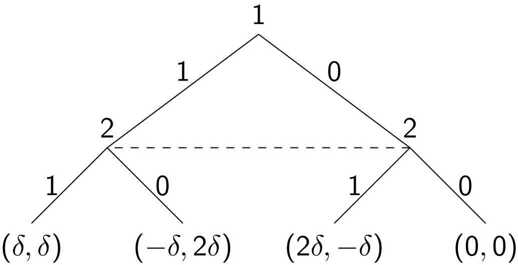

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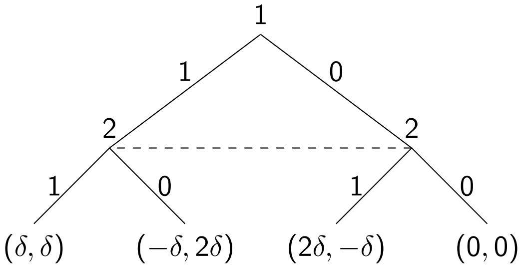
- The first subgame that we will analyze is the one that the players encounter after having play $(e_1^1 = 0, e_2^1 = 0)$ in $T = 1$:



The Nash equilibria can be seen by writing out the normal form of the game.

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	δ, δ	$-\delta, 2\delta$
$e_1 = 0$	$2\delta, -\delta$	$0, 0$

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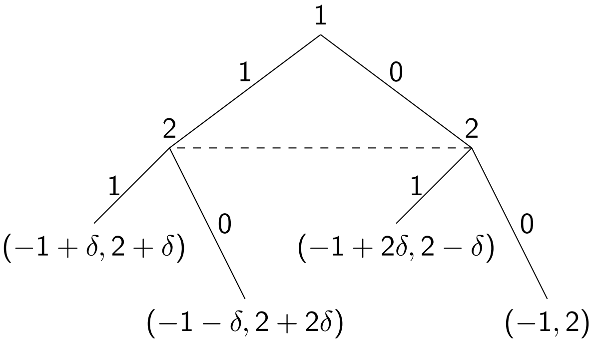
- Therefore after having observed $(e_1^1 = 0, e_2^1 = 0)$ in the first period, both players will play $(e_1^2 = 0, e_2^2 = 0)$ in period 2

Consider the subgame following a play of $(e_1^1 = 1, e_2^1 = 0)$ in the first period. The extensive form of this subgame is given by:



The normal form of this subgame can be seen in the Table

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1 + \delta, 2 + \delta$	$-1 - \delta, 2 + 2\delta$
$e_1 = 0$	$-1 + 2\delta, 2 - \delta$	$-1, 2$

- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium

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- In any SPNE, $(e_1^2 = 0, e_2^2 = 0)$ must be played after observing $(e_1^1 = 1, e_2^1 = 0)$

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames

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- Why is this the case?

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$

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- Why is this the case?

- The idea is that payoffs that have accrued in period 1 are essentially sunk, and have no influence on incentives in period 2

To see this consider the normal form representation in the subgame after the observation of $(e_1^1 = 1, e_2^1 = 0)$

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1+\delta, 2+\delta$	$-1-1\delta, 2+2\delta$
$e_1 = 0$	$-1+2\delta, 2-\delta$	$-1, 2$

- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by δ to obtain the following payoff matrix

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

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- We can do the same thing for player 2's payoffs and get the payoff matrix

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

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- Both players play ($e_1^2 = 0, e_2^2 = 0$) after any information set in the last period

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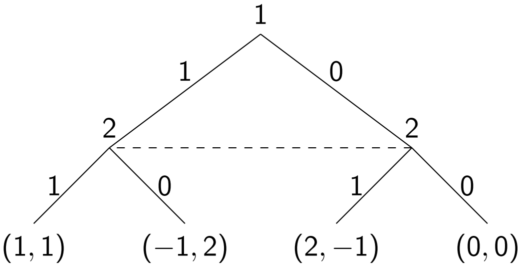
- Both players anticipate that $(e_1^2 = 0, e_2^2 = 0)$ will be played after any chosen action profile in the first period
- We can simplify the extensive form game to the following:



If we draw the normal form of this game, then we get:

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

The unique Nash equilibrium of the above normal form game is $(e_1^1 = 0, e_2^1 = 0)$

Therefore the unique SPNE is:

$$\left(\left(\begin{array}{cc} & e_1^2 = 0 \\ e_1^1 = 0 & e_1^2 = 0 \\ & e_1^2 = 0 \\ & e_1^2 = 0 \end{array}\right), \left(\begin{array}{cc} & e_2^2 = 0 \\ e_2^1 = 0 & e_2^2 = 0 \\ & e_2^2 = 0 \\ & e_2^2 = 0 \end{array}\right)\right)$$

In other words both players always shirk

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- Whenever the stage game has a unique NE, then the only SPNE of a **finite horizon** repeated game with that stage game is the repetition of the stage game NE

Theorem

Suppose that the stage game G_1 *has exactly one NE,* $(a_1^*, a_2^*, \ldots, a_n^*)$*. Then for any* $\delta \in (0, 1]$ *and any* T*, the* T*-times repeated game has a unique SPNE in which all players* i *play* a_i^* *at all information sets.*

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- Knowing that the stage game Nash equilibrium is going to be played tomorrow, at any information set, we can ignore the past payoffs
- We concentrate just on the payoffs in the future. Thus in period $T-1$, player i simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$

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- Following exactly this induction, we can conclude that every player must play a_i^* at all times and all histories