Lecture 10: Game Theory // Preliminaries

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Introduction

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- Although p is determined from the interaction of all agents (aggregate supply $=$ aggregate demand)

Definition (Strategic Interaction)

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- Originally, game theory was developed to design optimal strategies in games like chess or poker
- However, it allows to study a wide range of situations that were did not fit in traditional microeconomics theory

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- In 1967–1968, John Harsanyi formalized methods to study games of incomplete information
- In the 1970s, game theory became part of main stream economics (and other social sciences)

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- The information available to each player
- How the results of the game depends on the actions taken by each individual
- How individuals value the results of the game

Example (Matching pennies (pares y nones) – Sequential)

Two players, Ana & Bart, choose whether to show one or two fingers. First, Ana shows fingers to Bart, then Bart, after observing Ana's play, chooses how many fingers to show. If the total number of fingers is even, then Bart pays Ana 1,000 MXN. If the total number of fingers is odd, then Ana pays Bart 1,000 MXN.

A few examples

Example (Matching pennies (pares y nones) – Simultaneous)

Two players, Ana & Bart, choose whether to show one or two fingers simultaneously. If the total number of fingers is even, then Bart pays Ana 1,000 MXN. If the total number of fingers is odd, then Ana pays Bart 1,000 MXN.

- We assume agents maximize their **expected** utility

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  - Have a well defined utility function
  - Under uncertainty they maximize the expected utility

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- In other words, the specific utility function has important repercussions

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- All 3 agents have the "same preferences"

Utility	Lottery 1	Lottery 2
$\mathbb{E}u^1$	$0.5\ln(56) + 0.5\ln(46) \approx 3.92$	$0.5\ln(101) + 0.5\ln(1) \approx 2.3$
$\mathbb{E}u^2$	$0.5(56) + 0.5(46) = 51$	$0.5(101) + 0.5(1) = 51$
$\mathbb{E}u^3$	$0.5e^{56} + 0.5e^{4}6 \approx 1.04 \times 10^{24}$	$0.5e^{101} + 0.5e^{1} \approx 3.65 \times 10^{43}$

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- Proof that linear (or afine) transformations of the utility function represent the same preferences under uncertainty.

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- They go around for ever saying "pass"

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- What happens?
- The first two pass, the third says “white”
- Why?
- They already knew there was at least a white hat (they knew there were at least two)
- They already knew everyone knew there was at least a white hat
- Now they all now, that everyone knows, that everyone knows (ad infinitum) that there is a white hat.

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- We will always assume things are common knowledge (there are some extensions to the cases when utility functions are not common knowledge)

We will use the following notation:

- Game participants (players) will be denoted by index i, where $i = 1, .., N$ and there are N players.
- A_i is the space of possible actions for individual i. $a_i \in A_i$ is an action.
- If we have a vector $a = (a_1, ..., a_{i-1}, a_i, a_{i+1}, ..., a_N)$, then we will denote by $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_N)$ y $a = (a_i, a_{-i})$.
- S_i is the strategy space for individual i. $s_i \in S_i$ is a strategy.
- A strategy is a complete action plan. i.e., is an action for every possible contingency of the game a player may face.
- u^i is the utility of player i. $u_i(s_i, s_{-i})$, i.e., the utility of player i may depend on her strategy, as well as the strategy of other players.

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- For Bart, a strategy has an action for the two contingencies he may face (1) if Ana chooses 1 finger, (2) if Ana chooses 2 fingers
- $S_{Bart} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$