Lecture 15: Game Theory // Nash equilibrium

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Nash's Theorem

Dynamic Games

Theorem (Nash's Theorem)

Suppose that the pure strategy set S_i is finite for all players i. A Nash equilibrium always exists.

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  1. A Nash equilibrium is a fixed point of the best response functions
  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point
- Remember X^* is a fixed point of $F(X)$ if and only if $F(X^*) = X^*$

Proof - Part 1

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- $\Gamma(s_1^*, ..., s_n^*) = (s_1^*, ..., s_n^*)$
- Therefore $(s_1^*, ..., s_n^*)$ is a fixed point of Γ

Proof - Part 2

Theorem (Kakutani fixed-point theorem)

Let $\Gamma : \Omega \to \Omega$ *be a correspondence that is upper semi-continuous,* Ω *be non empty, compact (closed and bounded), and convex* $\Rightarrow \Gamma$ *has at least one fixed point*

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  - Therefore if $\Gamma(s_1, ..., s_n)$ has two images, those two images are connected (via all the mixed strategies that connect those two images)
- That happens to be the definition of upper semi-continous

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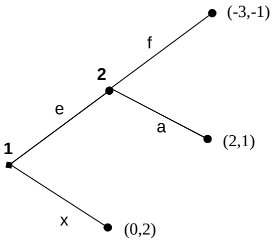
- Reminder: A (pure) strategy is a **complete contingent plan** of action at every information set
- The set of Nash equilibria of the extensive form game is simply the set of all Nash equilibria of the normal form representation of the game
- Some of the equilibria do not make much sense intuitively


(-3,-1)
f
2
e
a
(2,1)
1
x
(0,2)

	f	a
e	-3,-1	2,1
x	0,2	0,2

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Two Nash equilibria: (x,f) y (e,a).

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- In the previous example, f is not optimal if we reach the second period

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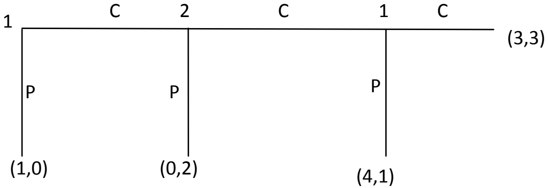
Theorem (Zermelo)

In every finite game where every information set has a single node (i.e., complete information), has an Nash equilibrium that can be derived via backwards induction. If the payouts to players are different in all terminal nodes, then the Nash equilibrium is unique.

Theorem (Zermelo II)

In any finite two-person game of perfect information in which the players move alternatingly and in which chance does not affect the decision making process, if the game cannot end in a draw, then one of the two players must have a winning strategy (i.e. force a win).

Centipede Game



	C	P
C,C	3,**3**	0,2
C,P	**4**,1	0,**2**
P,C	1,**0**	**1**,**0**
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- Nash equilibria are $\{(P, P), P\}$ and $\{(P, C), P\}$
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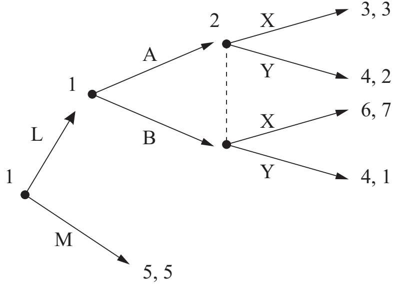
- But by backward induction, the solution is to play P in each period

Consider the following game



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- First, we need to defined a subgame

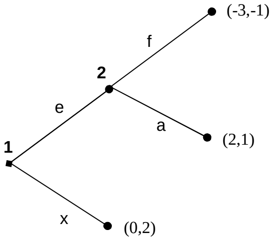
A sub-game, of a game in extensive form, is a sub-tree such that

- It starts in a single node
- If contains a node, it contains all subsequent nodes
- If it contains a node in an information set, it contains all nodes in the information set

Definition

A subgame of an extensive form game is the set of all actions and nodes that follow a particular node that is not included in an information set with another distinct node

By definition, the original game is a subgame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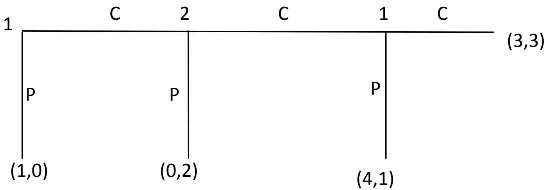
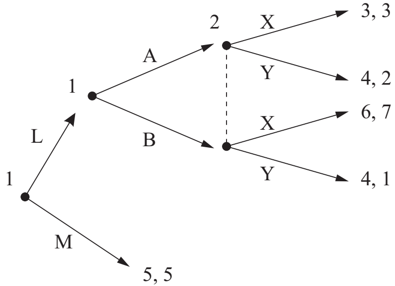




2
X
3, 3
A
Y
4, 2
1
X
6, 7
L
B
Y
4, 1
1
M
5, 5

Since in some games (where multiple nodes are in the same information set) we can't formally choose how people are optimizing, we extend the notion of backwards induction to subgames

Definition (Subgame perfect Nash equilibria)

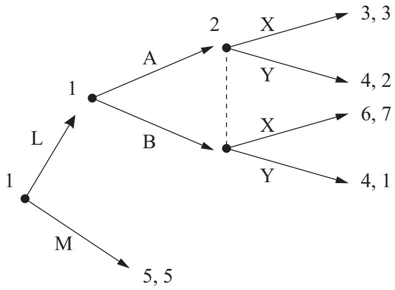
A pure strategy profile is a Subgame perfect Nash equilibria (SPNE) if and only if it involves the play of a NE in every subgame of the game.

Remark

Every SPNE is a NE

Remark

As in normal form games, mixed strategy SPNE can be defined but this is a bit technical. Thus, we will not worry about it for the purposes of the course.


2
X
3, 3
A
Y
4, 2
1
X
6, 7
L
B
Y
4, 1
1
M
5, 5

1 \ 2	X	Y
LA	3, 3	4, 2
LB	6, 7	4, 1
MA	5, 5	5, 5
MB	5, 5	5, 5

1 \ 2	X	Y
A	3, 3	4, 2
B	6, 7	4, 1

- The game has 3 NE: (LB,X), (MA,Y),(MB,Y)

- The subgame has a single NE: (B,X)

- The SPNE is (LB,X)