# Lecture 16

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Lecture16

## Lecture 16: Applications of Subgame Perfect Nash Equilibrium

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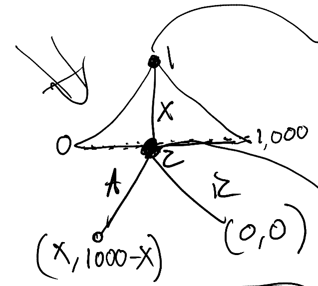
1. Player 1 makes a proposal $(x, 1000 - x)$ of how to split 1000 pesos among $(100, 900), \ldots, (800, 200), (900, 100)$
2. Player 2 accepts or rejects the proposal
3. If player 2 rejects both obtain 0. If 2 accepts, then the payoffs or the two players are determined by $(x, 1000 - x)$




$\Rightarrow S_1 = \{(x)\}$. 1 sola entrada $\in [0,1000]$

$\Rightarrow$ Infinitas contingencias $\Rightarrow$ Una estrategia de J2: cómo jugar dependiendo de X — una función de X.

$S_2$: A si $X < 1000$; $\{A, R\}$ si $X = 1{,}000$ — MR dinámica

$S_2 = \{$funciones q' van de $[0,1000]$ a $\{A, R\}\}$.

$S_2 = \{f : [0,1000] \to \{A, R\}\}$.

$J_1$: $X = 1{,}000$

ENPS = $\{(X = 1{,}000), (S_2(X) = \{ A \quad X < 1{,}000$

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- In any SPNE, player 1 makes the proposal $(900, 100)$

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- When extreme offers like $(900, 100)$ are made, player 2 rejects in many cases

$S_1 \quad x = 1{,}000$

$$EPS = \left\{ (S_1 = 1{,}000),\ \left( S_2(x) = \begin{cases} A & x < 1{,}000 \\ A & x = 1{,}000 \end{cases} \right) \right\}$$

$S_2 = \{ f : [0, 1000] \to \{A, R\} \}$.

UNA FUNCIÓN

$$\left( \left\{ S_1 = 1{,}000,\ S_2(x) = \begin{cases} A & x < 1{,}000 \\ R & x = 1{,}000 \end{cases} \right. \right)$$

NO ES EPS PUES $S_1$ SE QUIERE DESVIAR

EN ⇒ G' NO SEA EPS.

- This is far from what happens in reality
- When extreme offers like $(900, 100)$ are made, player 2 rejects in many cases
- Player 2 may care about inequality or positive utility associated with "punishment" aversion

## Lecture 16: Applications of Subgame Perfect Nash Equilibrium

Ultimatum Game

Alternating offers

Stackelberg Competition

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Ultimatum Game

Alternating offers

Stackelberg Competition

- Two players are deciding how to split a pie of size 1

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- The players would rather get an agreement today than tomorrow (i.e., discount factor)

- Player 1 makes an offer $\theta_1$

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- ... and on and on for $T$ periods
- If no offer is ever accepted, both payoffs equal zero

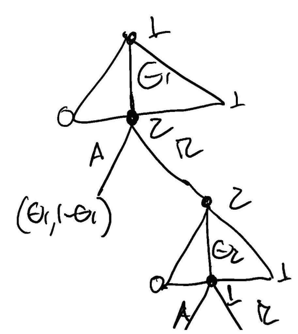
---

The discount factor is $\delta \leq 1$.
If Player 1 offer is accepted by Player 2 in round $m$,

$$\pi_1 = \delta^m \theta_m,$$

$$\pi_2 = \delta^m (1 - \theta_m).$$

If Player 2 offer is accepted, reverse the subscripts



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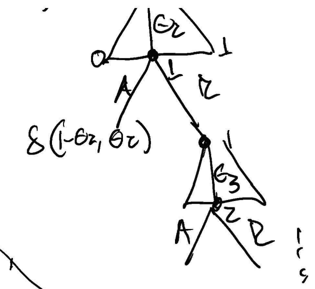


- Consider first the game without discounting

$\delta = 1$

Si ultimo, A o R

$S_1 = \begin{cases} A & \text{si } 1-\theta_T > 0 \\ \{A,R\} & \text{si } \theta_T = 1 \end{cases}$

T: $S_2 \rightarrow \theta_T = 1$

T-1: $S_2 = \begin{cases} A & \text{si } \theta_{T-1} = 0 \\ R & \text{si } \theta_{T-1} > 0 \end{cases}$

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- There is a unique SPNE: The player that makes the last offer gets the whole pie
- Last-mover advantage

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- Player 1 would accept (indifferent between accepting and rejecting) since the **whole pie** in the next period is worth $\delta$

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- Player 2 would accept since he can earn $(1-\delta)$ in the next period, which is worth $\delta(1-\delta)$ today

$\delta = 0.90$

T: $S_2 = \{A, R\} \Rightarrow \delta^T(1, 0)$ → EXPONENTE.
$S_1 = $ OFERTA

T−1: $S_1 = A$ ó $R \Rightarrow \delta^{T-1}(\theta_{T-1}, 1-\theta_{T-1})$ si ACEPTO $\Rightarrow A$ si $\delta^{T-1}\theta_{T-1} \geq \delta^T \cdot 1$, $U(T-1)$, $U(T)$; $\theta_{T-1} \geq \delta$
$S_2 = $ OFERTA → $\theta_{T-1} = \delta$
$\delta^{T-1}(\delta, 1-\delta)$

T−2: $S_2 = A$ ó $R \Rightarrow A$ PAGOS SON $\delta^{T-2}(1-\theta_{T-2}, \theta_{T-2}) \Rightarrow A$ si $\delta^{T-2}\theta_{T-2} \geq \delta^{T-1}(1-\delta)$, $U(T-2)$, $U(T-1)$; $\theta_{T-2} \geq \delta(1-\delta)$
$S_1 = $ OFRECE $\theta_{T-2} = \delta(1-\delta)$
PAGOS $\delta^{T-2}(1-\delta(1-\delta), \delta(1-\delta))$

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself
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- Player 1 would accept...
- ...
- In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting

Table 1 shows the progression of Player 1's shares when $\delta = 0.9$.

**Table 1: Alternating Offers over Finite Time**

| Round | 1's share | 2's share | Total value | Who offers? |
|---|---|---|---|---|
| $T-3$ | $\delta(1-\delta(1-\delta))$ | $1-\delta(1-\delta(1-\delta))$ | $\delta^{T-4}$ | 2 |
| $T-2$ | $1-\delta(1-\delta)$ | $\delta(1-\delta)$ | $\delta^{T-3}$ | 1 |
| $T-1$ | $\delta$ | $1-\delta$ | $\delta^{T-2}$ | 2 |
| $T$ | 1 | 0 | $\delta^{T-1}$ | 1 |

- If $T = 3$ (i.e, 1 offers, 2 offers, 1 offers)

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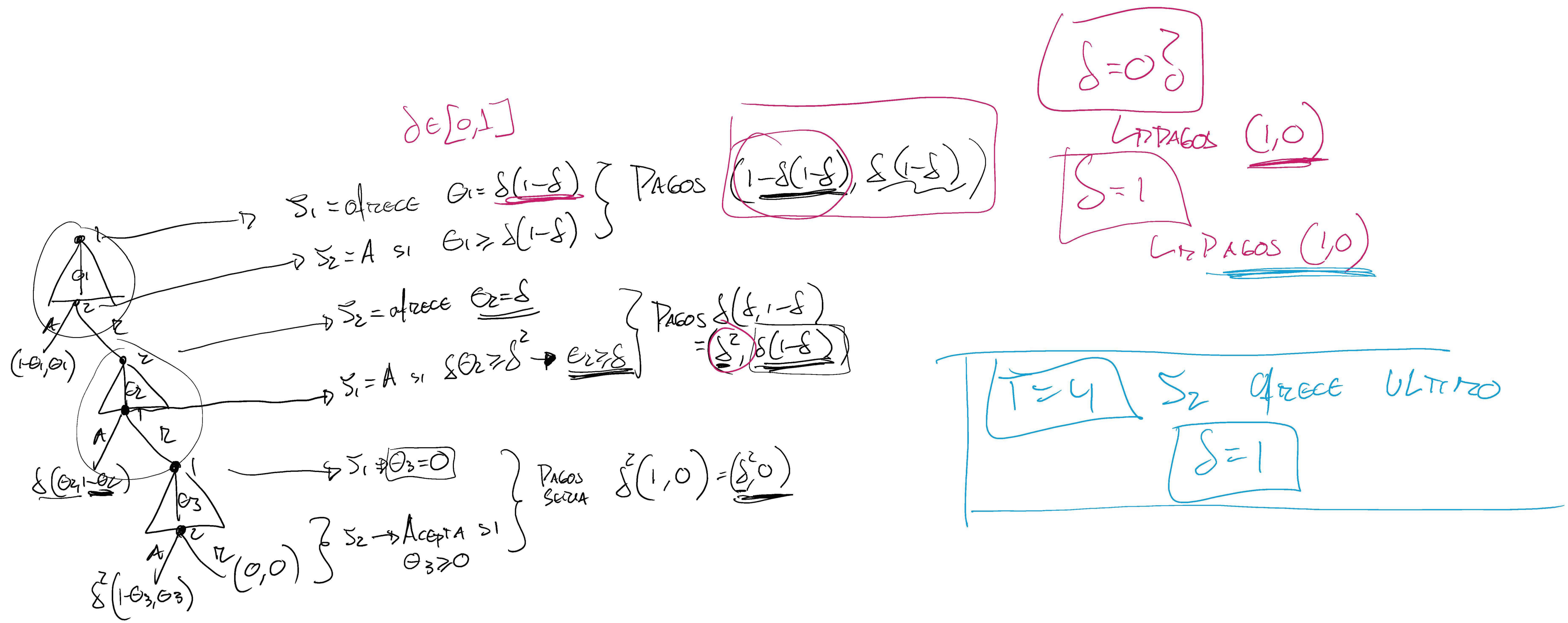
- One offers $\delta(1-\delta)$, 2 accepts in period 1

- Player 1 always does a little better when he makes the offer than when Player 2 does



- Player 1 always does a little better when he makes the offer than when Player 2 does
- If we consider just the class of periods in which Player 1 makes the offer, Player 1's share falls



Ultimatum Game

Alternating offers

Stackelberg Competition

• Factores Descuentos ≠

• 3 Jugadores

$$\delta = \frac{1}{1+r}$$



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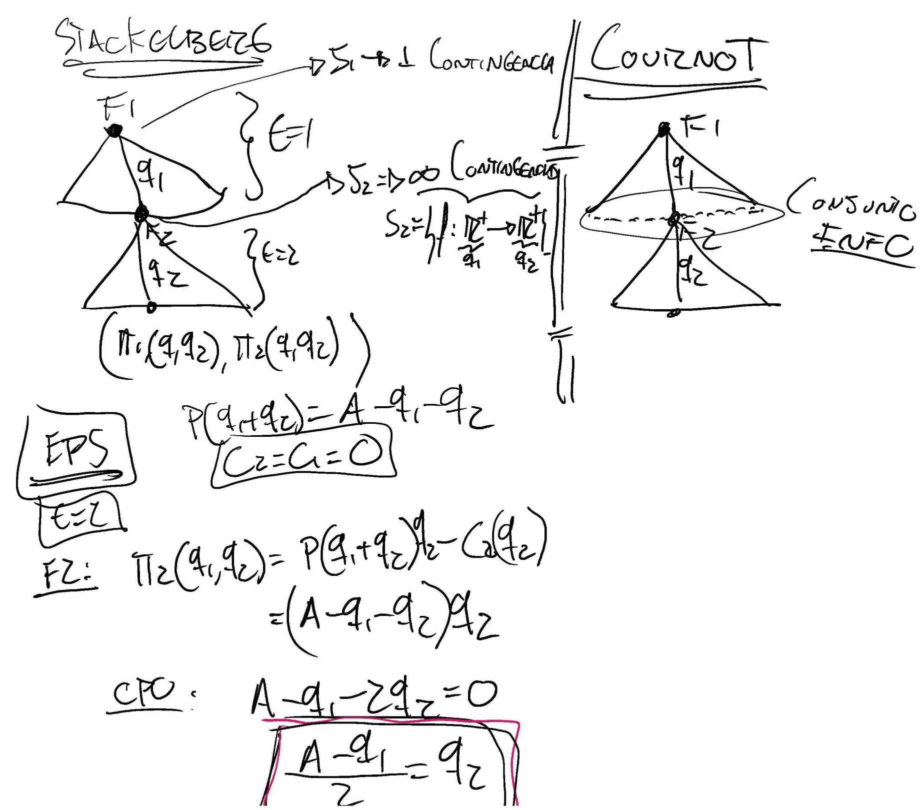
- Firms have the cost functions $c_i(q_i)$.

Te timing of the game is given by:

1. First Firm 1 chooses $q_1 \geq 0$
2. Second Firm 2 observes the chosen $q_1$ and then chooses $q_2$

- The game tree in this game is then depicted by an infinite tree

- Let us write down the normal form representation of this game.



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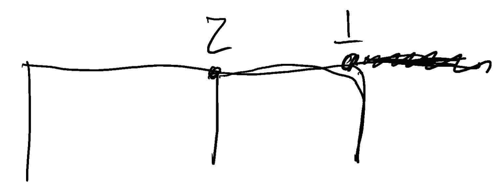
- A pure strategy for firm 1 is just a choice of $q_1 \geq 0$
- A strategy for firm 2 specifies what it does after every choice of $q_1$
- Firm 2's strategy is a function $q_2(q_1)$ which specifies exactly what firm 2 does if $q_1$ is the chosen strategy of player 1

The utility functions for firm $i$ when firm 1 chooses $q_1$ and firm 2 chooses the strategy (or function) $q_2(\cdot)$ is given by:

$$\pi_1(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_1 - c_1(q_1)$$
$$\pi_2(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_2(q_1) - c_2(q_2(q_1))$$

- There are many Nash equilibria of this game which are a bit counterintuitive

$$\frac{A-q_1}{2} = q_2$$




C=1

$$\pi_1(q_1,q_2) = P(q_1+q_2)q_1 - C_1(q_1)$$
$$= (A-q_1-q_2)q_1$$
$$= \left(A-q_1-\frac{A-q_1}{2}\right)q_1$$

FIRMA 1 PREVEE Q F2 ES RACIONAL.

CPO

$$\frac{\partial \pi_1}{\partial q_1} = A - 2q_1 - \frac{A}{2} + q_1 = 0$$

$$\frac{A}{2} = q_1$$

$$EPS = \left\{ S_1 = q_1 = \frac{A}{2},\ S_2 = q_2(q_1) = \frac{A-q_1}{2} \right\}$$

↳ ¿Cuales son las $q_1, q_2$ asociadas con el EPS?

$$q_1 = \frac{A}{2},\quad q_2 = \frac{A-(A/2)}{2} = \frac{A}{4}$$

$$EPS = \left\{ S_1 = q_1 = \frac{A}{2},\ S_2 = q_2 = \frac{A}{4} \right\}$$

F2 PRODUCE A/4 SIN IMPORTAR Q' HACE F1

$$\pi_1(q_1,q_2) = (A-q_1-q_2)q_1$$
$$= (A-q_1-A/4)q_1$$

CPO: $A - 2q_1 - \frac{A}{4} = 0$

$$\frac{3A}{4} = 2q_1$$

$$\frac{3A}{8} = q_1$$

$F_1$ SE QUIERE DESVIAR DE $A/2$ a $3A/8$

TAREA: ¿EN Q' NO SEA EPS?

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- Cconsider the following specific game with demand function given by:

$$P(q_1 + q_2) = A - q_1 - q_2.$$

- Let the marginal costs of both firms be zero
- Then the normal form simplifies:

$$u_1(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_1,$$
$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

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- Let $\alpha \in [0, A)$ and consider the following strategy profile:

$$q_1^* = \alpha, q_2^*(q_1) = \begin{cases} A & \text{if } q_1 \neq \alpha, \\ \frac{A-\alpha}{2} & \text{if } q_1 = \alpha. \end{cases}$$

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- Let us check that indeed this constitutes a Nash equilibrium

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- Firm 1 is best responding to player 2's strategy.

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- Thus, firm 2 wants to choose the optimal strategy $q_2(\cdot)$ that maximizes the following utility:

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- This would be the same outcome if firm 2 were the monopolist in this market

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- Firm 2 is threatening to overproduce if firm 1 produces anything at all
- As a result, the best that firm 1 can do is to produce nothing
- If firm 1 were to hypothetically choose $q_1 > 0$, then firm 2 would obtain negative profits if it indeed follows through with $q_2^*(q_1)$.

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- Lets continue with the setting in which marginal costs are zero and the demand function is given by $A - q_1 - q_2$

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- The utility function of firm 2 is given by:

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- So, player 2 solves:

$$\max_{q_2(\cdot)} (A - q_1 - q_2(q_1))q_2(q_1).$$

- **Case 1:** $q_1 > A$

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- In this case, the first order condition implies:

$$q_2^*(q_1) = \frac{A - q_1}{2}.$$

- Thus, in any SPNE, player 2 must play the following strategy:

$$q_2^*(q_1) = \begin{cases} \frac{A-q_1}{2} & \text{if } q_1 \leq A \\ 0 & \text{if } q_1 > A. \end{cases}$$

- Then player 1's utility function given that player 2 plays $q_2^*$ is given by:

$$u_1(q_1, q_2^*(\cdot)) = q_1(A - q_1 - q_2^*(q_1)) = \begin{cases} q_1(A - q_1) & \text{if } q_1 > A, \\ q_1 \frac{A - q_1}{2} & \text{if } q_1 \leq A. \end{cases}$$

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- Thus, firm 1 maximizes:

$$\max_{q_1 \in [0, A]} q_1 \frac{A - q_1}{2}.$$

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- The **equilibrium outcome** is for firm 1 to choose $A/2$ and firm 2 to choose $A/4$

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- Lets solve for the set of SPNE (which is the same as NE) in the Cournot game with the same demand function and same costs
- In this case, $(q_1^*, q_2^*)$ is a NE if and only if

$$q_1^* \in BR_1(q_2^*), q_2^* \in BR_2(q_1^*).$$

- For $q_1^* \in BR_1(q_2^*)$, we need $q_1^*$ to solve the following maximization problem:

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- As a result, solving these two equations, we get:

$$q_1^* = q_2^* = \frac{A}{3}.$$

In the Cournot game, note that firms' payoffs are:

$$\pi_1^c = \frac{A^2}{9}, \pi_2^c = \frac{A^2}{9}.$$

As we already saw, this was not Pareto efficient since each firm is getting a payoff that is strictly less than 1/2 of the monopoly profits.

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- But by choosing something optimal, firm 1 will be able to do even better