Lecture 17

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Lecture17

Lecture 17: Applications of Subgame Perfect Nash Equilibrium

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Lecture 17: Repeated Games

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- A game can repeat itself several times
- Static games turn into dynamic by repetition
- We will use (G, T) to denote that game G is repeated T times

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2. Players observe the actions chosen by the players in period 1. Then in period 2, players simultaneously play the game G.
3. This game proceeds until time T.
4. After time T, if the action profiles chosen in times $1, 2, \dots, T$ are given by $((a_i^1, a_{-i}^1), \dots, (a_i^T, a_{-i}^T))$:

$$U_{TOTAL} = \sum_{t=1}^{T} \delta^{t-1} u_i(a_i^t, a_{-i}^t).$$

Consider the following two-player game:

- Each player $i = 1, 2$ simultaneously decide whether to play $e_i = 1$ (work) or $e_i = 0$ (shirk)

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- Each player $i = 1, 2$ simultaneously decide whether to play $e_i = 1$ (work) or $e_i = 0$ (shirk)
- Working incurs a cost of 1 however increases the utility of the other player $-i$ by 2
- Thus,

$$u_i(e_i, e_{-i}) = 2e_{-i} - e_i.$$

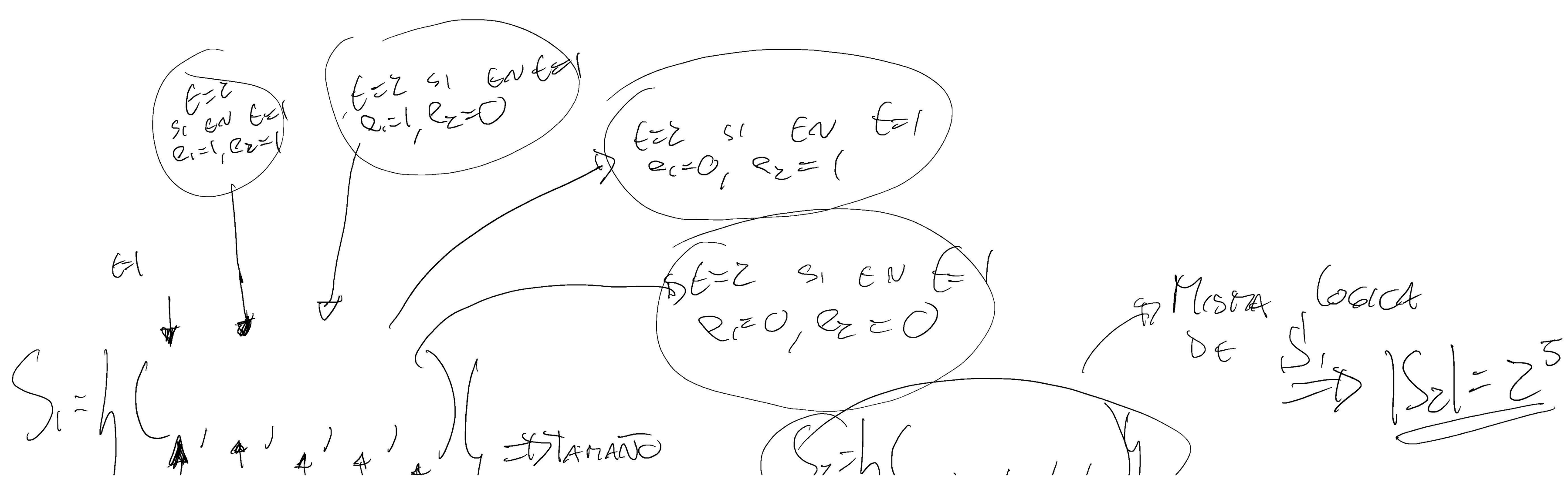
S_2

Prisoner's Dilemma (Game G)

S_1	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

$\to$ E.N $= (e_1 = 0, e_2 = 0)$

- What happens when $T = 1$



- What happens when δ = 1
- All Players 1 and 2 will both choose ($e_1 = 1, e_2 = 0$)

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1. Both players play the simultaneous move game G.
2. Both players observe the actions chosen by the two players. Then they play G again.
3. Then payoffs are realised as the discounted sum of the utilities of the actions in each period with discount factor δ ∈ (0, 1].

Suppose that the two players chose ($e_1 = 1, e_2 = 1$) in the first period. In the second period, they chose ($e_1 = 0, e_2 = 1$).

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$u_1 = 1 + δ \cdot 2$
$u_2 = 1 + δ(-1)$

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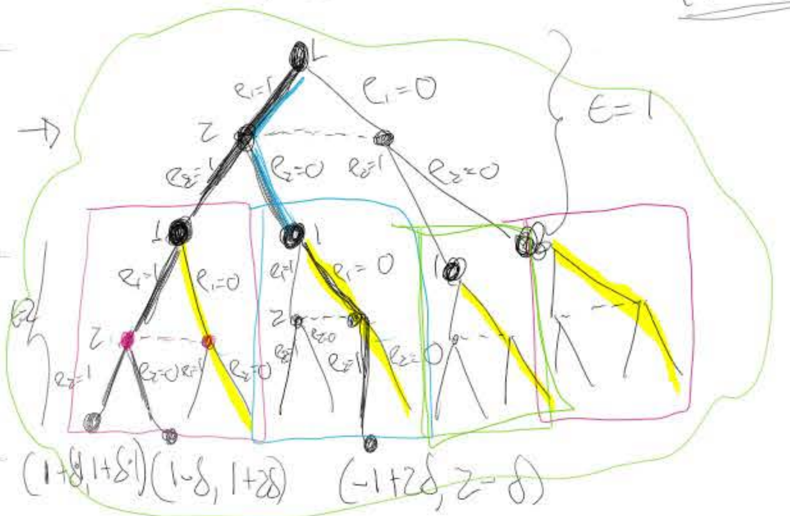
- A pure strategy for player 1 must specify what to do in each of these information sets.

$S_1 = \{(\ ,\ ,\ ,\ ,\ )\}$ ⟹ TAMAÑO $|S_1| = 2^5$

2 OPCIONES 2 2 2 2

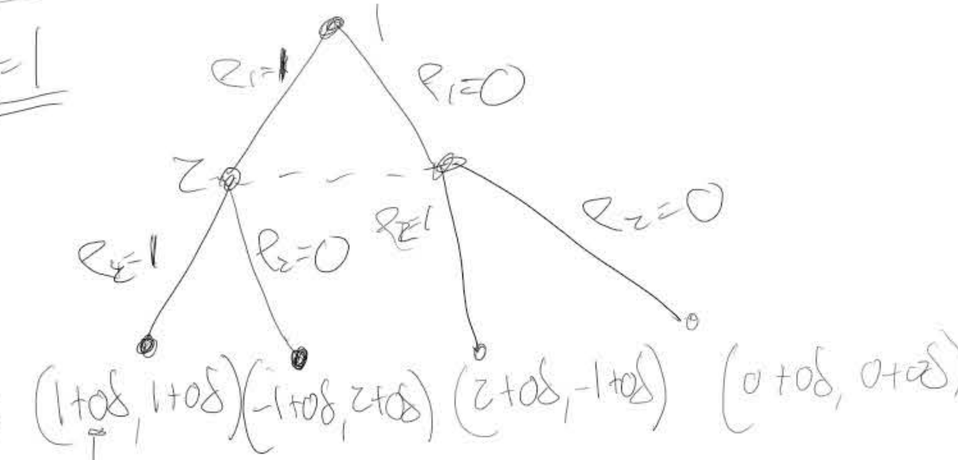
$S_2 = \{(\ ,\ ,\ ,\ ,\ )\}$



5 SUBJUEGOS

⟹ TENIENDO EN CUENTA EN EN t=2, DE LOS SUBJUEGOS

$t=1$



$t=2$ ⟶ 4 SUBJUEGOS PROPIOS
SON MUY PARECIDOS

DESDE PUNTO VISTA $t=1$ (TENIENDO EN CUENTA EN. $t=2$)

$t=2$ SUBJUEGO

	$e_2=1$	$e_2=0$
$e_1=1$	$U_{t=1}+δ(1,1)$	$U_{t=1}+δ(-1,2)$
$e_2=0$	$U_{t=1}+δ(2,-1)$	$U_{t=1}+δ(0,0)$

EQUIVALENTE

⟹ $U_{t=1}$ $(U^1_{t=1}, U^2_{t=1})$ + δ

(1,1)	(-1,2)
(2,-1)	(0,0)

⟶ MATRIZ

		S_2	
		$e_2=1$	$e_2=0$
S_1	$e_1=1$	1+0δ, 1+0δ	-1+0δ, 2+0δ
	$e_1=0$	2+0δ, -1+0δ	0+0δ, 0+0δ

$(0,0)δ$ + MATRIZ

1,1	-1,2
2,-1	0,0

PAGO EN. EN $t=2$

EN = $(e_1^{t=1}=0, e_2^{t=1}=0)$

SUBJUEGO #1 (ROSO IZQ)

$(1,1)+δ$ MATRIZ ⟹ EN = $(e_1^{t=2}=0, e_2^{t=2}=0)$

- We will solve for the set of pure SPNE of this game.
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 (2^5) pure strategies

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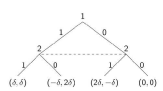
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 (2^5) pure strategies
- Similarly, player 2 has a total of 32 pure strategies

- There are 5 subgames



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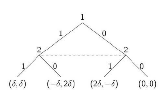
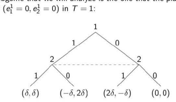
- Start at the end of the game (i.e., $T = 2$)



- There are 5 subgames
- Start at the end of the game (i.e., $T = 2$)
- The first subgame that we will analyze is the one that the players encounter after having play ($e_1^1 = 0, e_2^1 = 0$) in $T = 1$:



The Nash equilibria can be seen by writing out the normal form of the game.

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	δ, δ	$-\delta, 2\delta$
$e_1 = 0$	$2\delta, -\delta$	$0, 0$

- This game has a unique Nash equilibrium in which the players play ($e_1^2 = 0, e_2^2 = 0$)

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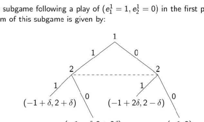
- Therefore after having observed ($e_1^1 = 0, e_2^1 = 0$) in the first period, both players will play ($e_1^2 = 0, e_2^2 = 0$) in period 2

Consider the subgame following a play of ($e_1^1 = 1, e_2^1 = 0$) in the first period. The extensive form of this subgame is given by:



$(1,1) + \delta$ MATRIZ $\Rightarrow EN = (e_1 \overset{t=2}{=} 0, e_2 \overset{t=2}{=} 0)$

SS#2 (AZUL)

$(-1,2) + \delta$ MATRIZ $\Rightarrow EN = (e_1 \overset{t=2}{=} 0, e_2 \overset{t=2}{=} 0)$

SS#3 (VERDE)

$(2,-1) + \delta$ MATRIZ $\Rightarrow EN = (e_1 \overset{t=2}{=} 0, e_2 \overset{t=2}{=} 0)$

SS#4 (DERECHA)

$(0,0) + \delta$ MATRIZ $\Rightarrow EN = (e_1 \overset{t=2}{=} 0, e_2 \overset{t=2}{=} 0)$

EPS = { t=1 NINGUNO COLABORA ; t=2 NINGUNO COLABORA } = { $S_1 = (\underset{e_1}{0}, \underset{e_1}{0}, \underset{e_1}{0}, \underset{e_1}{0}, \underset{e_1}{0})$, $S_2 = (\underset{e_2}{0}, \underset{e_2}{0}, \underset{e_2}{0}, \underset{e_2}{0}, \underset{e_2}{0})$ }

t=1 ; t=2 ; t=2

The normal form of this subgame can be seen in the Table

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1+\delta, 2+\delta$	$-1-\delta, 2+2\delta$
$e_1 = 0$	$-1+2\delta, 2-\delta$	$-1, 2$

- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium

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- In any SPNE, $(e_1^2 = 0, e_2^2 = 0)$ must be played after observing $(e_1^1 = 1, e_2^1 = 0)$

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames

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- Why is this the case?

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$
- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames
- Regardless of the observed action, $(0, 0)$ is played in period 2
- Why is this the case?
- The idea is that payoffs that have accrued in period 1 are essentially sunk, and have no influence on incentives in period 2

To see this consider the normal form representation in the subgame after the observation of $(e_1^1 = 1, e_2^1 = 0)$

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1+\delta, 2+\delta$	$-1-1\delta, 2+2\delta$
$e_1 = 0$	$-1+2\delta, 2-\delta$	$-1, 2$

- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by δ to obtain the following payoff matrix

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
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- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by δ to obtain the following payoff matrix
- We can do the same thing for player 2's payoffs and get the payoff matrix

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

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- This will be true no matter the action profile played in period 1

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- Basically after any history, the strategic normal form is essentially the same as the original prisoner's dilemma
- Both players play $(e_1^2 = 0, e_2^2 = 0)$ after any information set in the last period

- Now let us see what must be played in the first period by the two players

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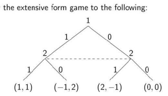
- Both players anticipate that $(e_1^2 = 0, e_2^2 = 0)$ will be played after any chosen action profile in the first period
- We can simplify the extensive form game to the following:



If we draw the normal form of this game, then we get:

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

The unique Nash equilibrium of the above normal form game is $(e_1^1 = 0, e_2^1 = 0)$

Therefore the unique SPNE is:

$$\left(\left(e_1^1 = 0 \quad \begin{matrix} e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \end{matrix} \right), \left(e_2^1 = 0 \quad \begin{matrix} e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \end{matrix} \right) \right)$$

In other words both players always shirk

- Here the unique SPNE requires all players to play $e_i = 0$ at all periods and all information sets

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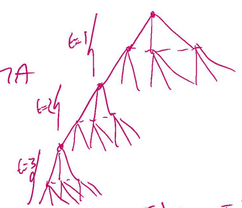
- Thus, the equilibrium outcome is simply the repetition of the unique NE of the stage game
- This holds more generally when the stage game has a unique NE
- Whenever the stage game has a unique NE, then the only SPNE of a **finite horizon** repeated game with that stage game is the repetition of the stage game NE

TEOREMA



Theorem

Suppose that the stage game G has exactly one NE $(a_1^*, a_2^*, \ldots, a_n^*)$. Then for any $\delta \in (0, 1]$ and any T, the T-times repeated game has a unique SPNE in which all players i play a_i^* at all information sets.

T: $U_T = U_{T-1} + \delta^{T-1} U(a_i^T, a_{-i}^T)$

SUMA PAGOS EN TODO PERIODO ANTERIOR — NO DEPENDE DE a_i Y a_{-i}

INCENTIVOS SON LOS MISMOS JUEGO BASE PUES "NO HAY FUTURO" Y EL "PASADO ES PASADO" — $U_{T-1} + \delta^{T-1}$ MATRIZ G

⇒ EN T SE JUEGA EL E.N.

- The basic idea of the proof for this proposition is exactly the same that we saw in the repeated prisoner's dilemma

$T-1$: $U = U_{T-2} + \delta^{T-2} U(a_i^{T-1}, a_{-i}^{T-1}) + \delta^{T-1} U(E.N.)$

UTILIDAD EN ULTIMO PERIODO

YA PASO ANTES — "PASADO ES PASADO", "FUTURO ESTÁ DETERMINADO" — JUEGO E.N. DE G

$\to U_{T-2} + \delta^{T-2}$ MATRIZ G $+ \delta^{T-1} U(E.N.)$

INCENTIVOS SON LOS MISMOS A G ⇒ JUGAMOS E.N. G

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- All past payoffs are sunk

$T-2$

$U = U_{T-3} + \delta^{T-3} U(a_i^{T-2}, a_{-i}^{T-2}) + \delta^{T-2} U(EN) + \delta^{T-1} U(EN)$

- The basic idea of the proof for this proposition is exactly the same that we saw in the repeated prisoner's dilemma
- All past payoffs are sunk
- In the last period, the incentives of all players are exactly the same as if the game were being played once

⇒ JUEGO E.N. G

- The basic idea of the proof for this proposition is exactly the same that we saw in the repeated prisoner's dilemma
- All past payoffs are sunk
- In the last period, the incentives of all players are exactly the same as if the game were being played once
- Thus all players must play the stage game Nash equilibrium action regardless of the history of play up to that point

$t=1 \to U = U(a_i^{t=1}, a_{-i}^{t=1}) + \left(\sum_{t=2}^{T} \delta^{t-1} U(E.N.)\right)$

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MATRIZ G $+ \sum_{t=2}^{T} \delta^{t-1} U(E.N.)$

DETERMINISTICO

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- Knowing that the stage game Nash equilibrium is going to be played tomorrow, at any information set, we can ignore the past payoffs
- We concentrate just on the payoffs in the future. Thus in period $T-1$, player i simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$

⇒ JUEGO E.N. DE G

- What player i plays today has no consequences for what happens in period T since we saw that all players will play a^* no matter what happens in period $T-1$

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \ldots, a_n^{T-1} = a_n^*$.

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \ldots, a_n^{T-1} = a_n^*$.

- Following exactly this induction, we can conclude that every player must play a_i^* at all times and all histories