Lecture 19

Thursday, May 07, 2020 3:46 PM

Lecture19...

Lecture 19: Infinitely Repeated Games

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Introduction to Infinitely Repeated Games

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- One of the features of **finitely** repeated games was that if the stage game had a **unique** Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium
- This happened because there was a last period from which we could induct backwards (and there was a domino effect!)

- When the game is instead **infinitely** repeated, this argument no longer applies since there is no such thing as a last period

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- In each period $t = 0, 1, 2, \ldots$, players simultaneously choose an action $a_i \in A_i$ and the chosen action profile $(a_1, a_2, \ldots, a_n)$ is observed by all players
- Then play moves to period $t + 1$ and the game continues in the same manner.

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- We can represent each information set of player i by a history:

$$h^0 = (\emptyset), h^1 = (a^0 := (a_1^0, \ldots, a_n^0)), \ldots, h^t = (a^0, a^1, \ldots, a^{t-1})$$

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- We denote the set of all histories at time t as $\boxed{H^t}$

Prisoner's Dilemma

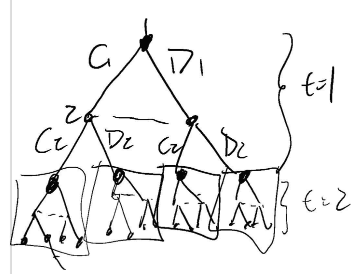
	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$0, 0$

$EN = (D_1, D_2)$

- For example, if the stage game is the prisoner's dilemma, at period ~~1~~ 2, there are 4 possible histories:

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^{\cancel{1}2}.$$

$H^1 = \{\emptyset\}$



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- For time t, H^t consists of 4^t possible histories
- This means that there is a **one-to-one** mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree
- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player i in each time t

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Q' HACER DEPENDIENDO
HISTORIA JUEGO.

- Intuitively, s_i describes exactly what player i would do at every possible history h^t, where $s_i(h^t)$ describes what player i would do at history h^t

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- There can be more complicated strategies such as the following:

$$s_i(h^t) = \begin{cases} C_i & \text{if } t = 0 \text{ or } h^t = (C, C, \ldots, C), \\ D_i & \text{otherwise.} \end{cases}$$

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$S_1 \to$ DESENCADENANTE

	S_1	S_2
1	C_1	C_2
2	C_1	C_2
3	C_1	C_2
4	C_1	D_2
5	D_1	
	D_1	
	D_1	
	⋮	
	D_1	

- The above is called a **grim trigger strategy**

- How are payoffs determined in the repeated game?

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- Then the payoff of player i in this repeated game is given by:
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- Intuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

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- Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain

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- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all $i = 1, 2$ and all h^t.

$$V_1 = \sum_{t=1}^{\infty} \delta^{t-1} \underbrace{u(C_1, C_2)}_{=1} = \sum_{t=1}^{\infty} \delta^{t-1} = 1 + \delta + \delta^2 + \dots + \dots = \boxed{\frac{1}{1-\delta}}$$

$$S_T = 1 + \cancel{\delta} + \cancel{\delta^2} + \dots + \cancel{\delta^T}$$

$$-\delta S_T = -\cancel{\delta} - \cancel{\delta^2} + \dots - \delta^{T+1}$$

$$S_T(1-\delta) = 1 - \delta^{T+1}$$

$$\to S_T = \frac{1-\delta^{T+1}}{1-\delta}$$

$$\lim_{T\to\infty} S_T = \frac{1}{1-\delta}$$

- Lets see some examples of how to compute payoffs in the repeated game
- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all $i = 1, 2$ and all h^t.
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- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \ldots)$

$t=0 \rightarrow (C_1, C_2)$
$t=1 \rightarrow (C_1, C_2)$
$t=2 \rightarrow (C_1, C_2)$
$\vdots$

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- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \ldots)$
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

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- Suppose that $s_i(h^0) = \boxed{(C_1, D_2)}$ and the strategy profile says to do exactly what the opponent did in the previous period

Ojo por ojo (TIT FOR TAT)

$t=0 \to (C_1, D_2)$
$t=1 \to (D_1, C_2)$
$t=2 \to (C_1, D_2)$
$t=3 \to (D_1, D_2)$

- How about a more complicated strategy profile?
- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
- Then if both players play these strategies, then the sequence of actions that arise is:

$$(C_1, D_2), (D_1, C_2), (C_1, D_2), \ldots$$

U_1

$$\underbrace{(-1)}_{U(C_1,D_2)}^{t=0} + \delta \underbrace{2}_{U(D_1,C_2)}^{t=1} + \delta^2 \underbrace{(-1)}_{U(C_1,D_2)}^{t=2} + \delta^3 \underbrace{(2)}_{U(D_1,C_2)}^{t=3} + \ldots$$

$$(-1)\underbrace{\left[1+\delta^2+\delta^4+\delta^6+\ldots\right]} + 2\left[\delta+\delta^3+\delta^5+\ldots\right]$$

$$(-1)\frac{1}{1-\delta^2} + 2\delta\left(1+\delta^2+\delta^4+\delta^6+\ldots\right)$$

$$(-1)\frac{1}{1-\delta^2} + 2\delta \cdot \frac{1}{1-\delta^2}$$

$$\frac{2\delta-1}{1-\delta^2}$$

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$$(C_1, D_2), (D_1, C_2), (C_1, D_2), \ldots$$

- Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \boxed{\frac{2\delta-1}{1-\delta^2}}.$$

Lecture 19: Infinitely Repeated Games

U_2

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- That is a strategy profile $s = (s_1, \ldots, s_n)$ is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s_i' from time $t+1$ on

- This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s_i'.

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- We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

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- What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D_1 and D_2 always at all information sets
- Why is this a SPNE?
- We can use the one-stage deviation principle

$D_1 \quad D_1 \quad D_1 \quad \dots \quad D_1 \quad D_1 \dots$

$t=0 \quad t=1 \quad t=2 \quad \dots \quad t \quad t+1$

$D_2 \quad D_2 \quad D_2 \quad \dots \quad D_2 \quad D_2 \dots$

$U_1(\text{No Desviarse}) = 0 + 0\delta + 0\delta^2 + \dots 0\delta^t + 0\delta^{t+1} + \dots = 0$

$U_1(\text{Desviarse un periodo}) = 0 + 0\delta + 0\delta^2 + \dots + U(C_1, D_2)\delta^t + 0\delta^{t+1} + \dots = -1\delta^t$

$\to$ Periodo Desvie

↳ Me desvie solo un periodo

Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
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- Under this strategy profile s_1^*, s_2^*, for all histories h^t,

$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

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- Thus, for all histories h^t,

$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

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- Thus, (s_1^*, s_2^*) is a SPNE

In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^* be a Nash equilibrium of the stage game. Then the strategy profile s^* in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

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- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium
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- Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \ldots, C) \\ D_i & \text{if } h^t \neq (C, C, \ldots, C). \end{cases}$$

SI S_1 Y S_2 → JUEGA UNA ESTRATEGIA GATILLO (DESENCADENANTE) (¿EPS?)

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- The *equilibrium path of play* for this SPNE is for players to play C in every period

CASO 1: YA HUBO DESVIO

$$\boxed{h_t \neq (C, C, C, \ldots, C)}$$

$$V_t(\text{No Desviarme}) = \underbrace{V_{t-1}}_{\text{PASADO}} + \underline{U(D_1, D_2) + U(D_1, D_2)\delta + \cdots}$$

$$V_{t-1} + U(D_1, D_2)/1-\delta$$

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- How do we show that the above is indeed an SPNE?

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- How do we show that the above is indeed an SPNE?
- We use the one-stage deviation principle again
- We need to check the one-stage deviation principle at every history h^t.

Case 1:

- Suppose first that $h^t \neq (C, C, \ldots, C)$

CASO I: $\boxed{h_t \neq (C, C, C, \ldots, C)}$

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$$V_t(\text{No Desv}) = \frac{U(D_1, D_2)}{1-\delta} = \frac{0}{1-\delta} = 0$$

$$V_t(\text{Desv}) = U(C_1, D_2) + \delta U(D_1, D_2) + \delta^2 U(D_1, D_2) + \ldots$$

$$= U(C_1, D_2) + \delta U(D_1, D_2)\left[1 + \delta^2 + \delta^3 + \ldots\right]$$

$$= U(C_1, D_2) + \frac{\delta U(D_1, D_2)}{1-\delta}$$

$$= (-1) + \frac{\delta \cdot 0}{1-\delta} = -1$$

Prefiere No Desv ✓

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- So the above inequality is satisfied if and only if

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CASO NO DESVIO ANTES

$$V_t(\text{No DesV}) = U_1(C_1, C_2) + \delta U_1(C_1, C_2) + \delta^2 U_1(C_1, C_2) + \dots$$
$$= \frac{U_1(C_1, C_2)}{1-\delta} = \frac{1}{1-\delta}$$

$$V_t(\text{DesV}) = U_1(D_1, C_2) + \delta U_1(D_1, D_2) + \delta^2 U_1(D_1, D_2) + \dots$$
$$= U_1(D_1, C_2) + \delta U_1(D_1, D_2)\left[1 + \delta + \delta^2 + \dots\right]$$
$$= \underbrace{U_1(D_1, C_2)} + \frac{\delta U_1(D_1, D_2)}{1-\delta}$$
$$= 2 + \frac{\delta \cdot 0}{1-\delta} = 2$$

$$U(\text{No DesV}) \geq U(\text{DesV})$$

$$\geq u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

- But since $h^t \neq (C, C, \ldots, C)$, $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.
- So the above inequality is satisfied if and only if

$$u_i(D_i, D_{-i}) \geq u_i(C_i, D_{-i}).$$

- But this is satisfied since D is a Nash equilibrium of the stage game

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- In this case,

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$$= 1, V_i(s^* \mid (h^t, (D_i, C_{-i}))) = u_i(D) = 0.$$

$U(\text{No Desv}) \geq U(\text{Desv})$

$\frac{1}{1-\delta} \geq 2$

$\frac{1}{2} \geq 1-\delta$

$\delta \geq 1/2$

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- Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \geq 1/2$.

- The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^* is a Nash equilibrium of the stage game. Suppose that $\hat{a}$ is an action profile of the Nash equilibrium such that

$$\boxed{u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).}$$

Then there is some $\boxed{\delta^* < 1}$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play $\hat{a}$ in every period.

$\boxed{T = \infty}$