Lecture 3: General Equilibrium

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Lecture 3: General Equilibrium

Competitive equilibrium

Examples: Cobb-Douglas

Examples: Perfect Complements

Examples: Perfect Substitutes

Competitive equilibrium

Examples: Cobb-Douglas

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Examples: Perfect Substitutes

Hidden assumptions

- There is a market for each good
- Every agent can access the market without any cost
- There is a unique price for each good and all consumers know this price
- Each consumer can sell her initial endowment in the market and use the income to buy goods and services
- Consumers seek to maximize their utility given their budget restriction, independently of what everyone else is doing.
  - There is no centralized mechanism
  - People may not know others preferences or endowments
- There is perfect competition (i.e., everyone is a price taker)
- The only source of information agents are prices

Competitive equilibrium - Definition

Definition

A pair of an allocation and a price vector, $(x^*, p = (p_1, \ldots, p_L))$ is called a competitive equilibrium if the following conditions hold:

1. For all consumers $i = 1, 2, \ldots, I$, $x^{i*} = (x_1^{i*}, \ldots, x_L^{i*})$ solves the following maximization problem:

$$\max_{x^i} u_i(x^i)$$

$$\text{such that } p \cdot x^i \leq p \cdot \omega^i = \sum_{\ell=1}^{L} p_\ell \omega_\ell^i.$$

2. Markets clear: For each commodity $\ell = 1, 2, \ldots, L$, the following equation holds:

$$\sum_{i=1}^{I} x_\ell^{i*} = \sum_{i=1}^{I} \omega_\ell^i.$$

Competitive equilibrium - Properties

Remark

Suppose that at least one consumer has **strictly** *monotone preferences. Then if* (x^*, p) *is a competitive equilibrium,* $p_1, p_2, \ldots, p_L > 0$.

Remark

Suppose that at least one consumer has **weakly** *monotone preferences. Then if* (x^*, p) *is a competitive equilibrium, there for at least one* ℓ, $p_\ell > 0$.

Remark

If (x^*, p) *is a competitive equilibrium, then* (x^*, cp) *for* $c \in \mathbb{R}_+ +$ *is also a competitive equilibrium.*

Competitive equilibrium - Walras' Law

Theorem (Walras' Law)

Suppose that consumer i has weakly monotone preferences and that $\hat{x}^i \in x^{i*}(p)$. Then

$$p \cdot \hat{x}^i = \sum_{\ell=1}^{L} p_\ell \hat{x}^i_\ell = \sum_{\ell=1}^{L} p_\ell \omega^i_\ell = p \cdot \omega^i.$$

Theorem (Walras' Law - II)

Suppose that utility functions are **weakly** *monotonic. Suppose that $p = (p_1, \dots, p_L)$ is such that $p_L > 0$. Take any (x^*, p) in which Condition 1 holds for each consumer $i = 1, 2, \dots, I$ and markets clear for all commodities $\ell = 1, 2, \dots, L-1$. Then the market clearing condition will hold for commodity L as well.*

Walras' Law - proof

- For each consumer i, we must

$$\sum_{\ell=1}^{L} p_\ell x_\ell^{i*} = \sum_{\ell=1}^{L} p_\ell \omega_\ell^i.$$

Walras' Law - proof

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$$\sum_{\ell=1}^{L} p_\ell x_\ell^{i*} = \sum_{\ell=1}^{L} p_\ell \omega_\ell^i.$$

- If we sum the above across all I consumers, then we get:

$$\sum_{i=1}^{I} \sum_{\ell=1}^{L} p_\ell x_\ell^{i*} = \sum_{i=1}^{I} \sum_{\ell=1}^{L} p_\ell \omega_\ell^i.$$

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- Re-arranging:

$$\sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell x_\ell^{i*} = \sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell \omega_\ell^i.$$

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- Re-arranging:

$$\sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell x_\ell^{i*} = \sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell \omega_\ell^i.$$

- Re-arranging:

$$\sum_{\ell=1}^{L} p_\ell \sum_{i=1}^{I} \left(x_\ell^{i*} - \omega_\ell^i \right) = 0.$$

Walras' Law - proof

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Walras' Law - proof

- $$\sum_{\ell=1}^{L} p_\ell \sum_{i=1}^{I} \left(x_\ell^{i\,*} - \omega_\ell^i \right) = 0.$$

- $$p_L \sum_{i=1}^{I} \left(x_L^{i\,*} - \omega_L^i \right) = 0.$$

Walras' Law - proof

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Cobb-Douglas

$$u_A(x, y) = x^\alpha y^{1-\alpha}$$
$$u_B(x, y) = x^\beta y^{1-\beta}$$

Suppose

$$\alpha = 0.5$$
$$\beta = 0.5$$
$$\omega^A = (1.5, 0.5)$$
$$\omega^B = (0.5, 1.5)$$

Cobb-Douglas

Each individual solves

$$max \sqrt{x^i y^i}$$

s.t.

$$p_x x^i + p_y y^i \leq p_x w_x^i + p_y w_y^i$$

Cobb-Douglas

Each individual solves

$$max \sqrt{x^i y^i}$$

s.t.

$$p_x x^i + p_y y^i \leq p_x w_x^i + p_y w_y^i$$

We can set up a Lagrangean:

$$\mathcal{L} = \sqrt{x^i y^i} + \lambda \left(p_x w_x^i + p_y w_y^i - p_x x^i - p_y y^i \right)$$

Cobb-Douglas

Each individual solves

$$max \sqrt{x^i y^i}$$

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We can set up a Lagrangean:

$$\mathcal{L} = \sqrt{x^i y^i} + \lambda \left(p_x w_x^i + p_y w_y^i - p_x x^i - p_y y^i \right)$$

The FOC are:

$$\frac{1}{2}\sqrt{\frac{y^i}{x^i}} = \lambda p_x$$

$$\frac{1}{2}\sqrt{\frac{x^i}{y^i}} = \lambda p_y$$

Cobb-Douglas

Thus,

$$\frac{y^i}{x^i} = \frac{p_x}{p_y}$$

$$y^i = x^i \frac{p_x}{p_y}$$

Cobb-Douglas

Thus,

$$\frac{y^i}{x^i} = \frac{p_x}{p_y}$$

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We haven't used the budget restriction!

Cobb-Douglas

Thus,

$$\frac{y^i}{x^i} = \frac{p_x}{p_y}$$

$$y^i = x^i \frac{p_x}{p_y}$$

We haven't used the budget restriction!

$$p_x x^i + p_y y^i = p_x w_x^i + p_y w_y^i$$

$$p_x x^i + p_y x^i \frac{p_x}{p_y} = p_x w_x^i + p_y w_y^i$$

$$x^i = \frac{w_x^i p_x + w_y^i p_y}{2p_x}$$

$$y^i = \frac{w_x^i p_x + w_y^i p_y}{2p_y}$$

Cobb-Douglas

$$x^A = \frac{1.5p_x + 0.5p_y}{2p_x}$$

$$y^A = \frac{1.5p_x + 0.5p_y}{2p_y}$$

$$x^B = \frac{0.5p_x + 1.5p_y}{2p_x}$$

$$y^B = \frac{0.5p_x + 1.5p_y}{2p_y}$$

Now we can use condition 2 (market clear)

Cobb-Douglas

$$x^A = \frac{1.5p_x + 0.5p_y}{2p_x}$$

$$y^A = \frac{1.5p_x + 0.5p_y}{2p_y}$$

$$x^B = \frac{0.5p_x + 1.5p_y}{2p_x}$$

$$y^B = \frac{0.5p_x + 1.5p_y}{2p_y}$$

Now we can use condition 2 (market clear)

$$x^A + x^B = 2$$

$$y^A + y^B = 2$$

Cobb-Douglas

$$\frac{1.5p_x + 0.5p_y}{2p_x} + \frac{0.5p_x + 1.5p_y}{2p_x} = 2$$

$$\frac{p_x}{p_y} = 1$$

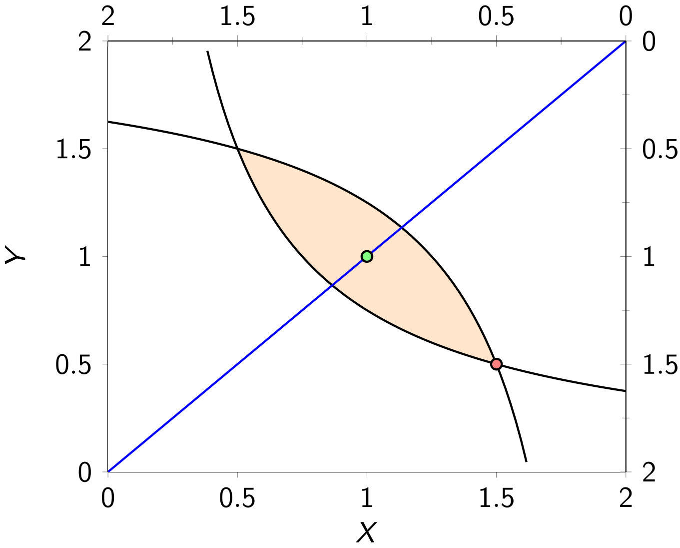
Cobb-Douglas

$$\frac{1.5p_x + 0.5p_y}{2p_x} + \frac{0.5p_x + 1.5p_y}{2p_x} = 2$$

$$\frac{p_x}{p_y} = 1$$

$$x^A = x^B = y^A = y^B = 1$$

Cobb-Douglas



Lecture 3: General Equilibrium

Competitive equilibrium

Examples: Cobb-Douglas

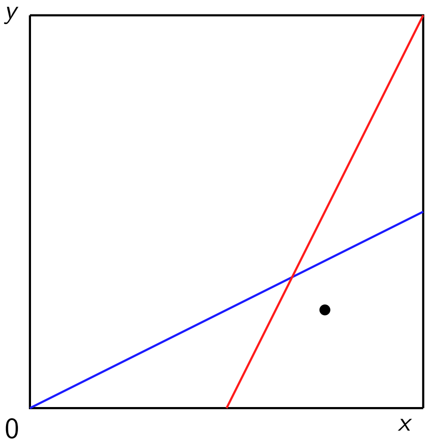
Examples: Perfect Complements

Examples: Perfect Substitutes

Perfect complements

Suppose that

$$u_A(x^A, y^A) = \min(x^A, 2y^A)$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (3, 1)$$
$$\omega^B = (1, 3)$$


y
0
x

Perfect complements

At a given price vector, consumer A can buy any combination (x^A, y^A) such that:

$$p_x w_x^A + p_y w_y^A \geq p_x x^A + p_y y^A$$

Perfect complements

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$$p_x w_x^A + p_y w_y^A \geq p_x x^A + p_y y^A$$

or equivalently

$$y^A \leq \frac{p_x w_x^A + p_y w_y^A}{p_y} - \frac{p_x}{p_y} x^A$$

Perfect complements

At a given price vector, consumer A can buy any combination (x^A, y^A) such that:

$$p_x w_x^A + p_y w_y^A \geq p_x x^A + p_y y^A$$

or equivalently

$$y^A \leq \frac{p_x w_x^A + p_y w_y^A}{p_y} - \frac{p_x}{p_y} x^A$$

How does this looks in the Edgeworth box?

If $\frac{p_x}{p_y} \neq 1$ Then, we will have the following restriction:

$$y^A \leq \frac{p_x}{p_y}\left(w_x^A - x^A\right) + w_y^A$$

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$$y^A \leq \frac{p_x}{p_y}\left(w_x^A - x^A\right) + w_y^A$$

Thus, replacing the values of w_x^A and w_y^A, we have:

$$y^A \leq \frac{p_x}{p_y}\left(3 - x^A\right) + 1$$

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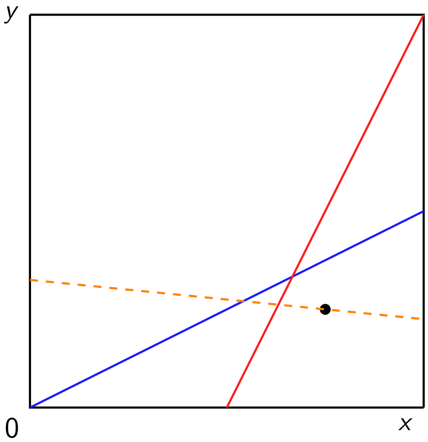
Thus, replacing the values of w_x^A and w_y^A, we have:

$$y^A \leq \frac{p_x}{p_y}\left(3 - x^A\right) + 1$$

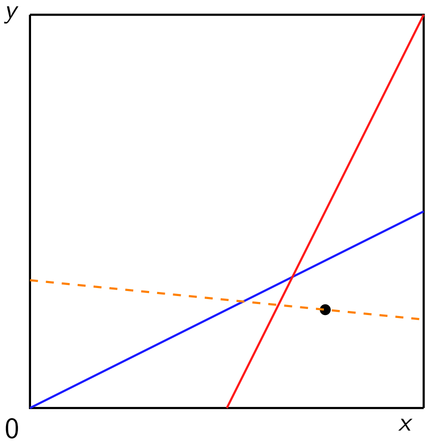
Note that, for the case $\frac{p_x}{p_y} = 1$, we have the following restriction:

$$y^A \leq 4 - x^A$$

$\frac{p_x}{p_y} < 1$

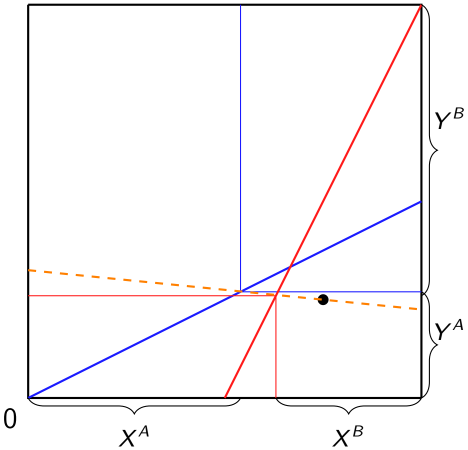


$\frac{p_x}{p_y} < 1$



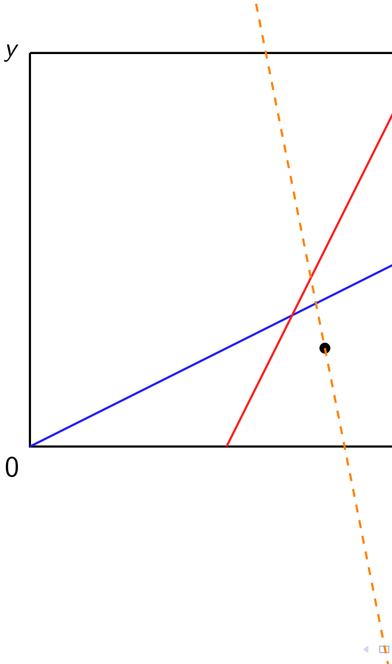
A can buy whats below the orange line, *B* what is above

$\frac{p_x}{p_y} < 1$



Excess demand of Y and excess supply of X

$\frac{p_x}{p_y} > 1$



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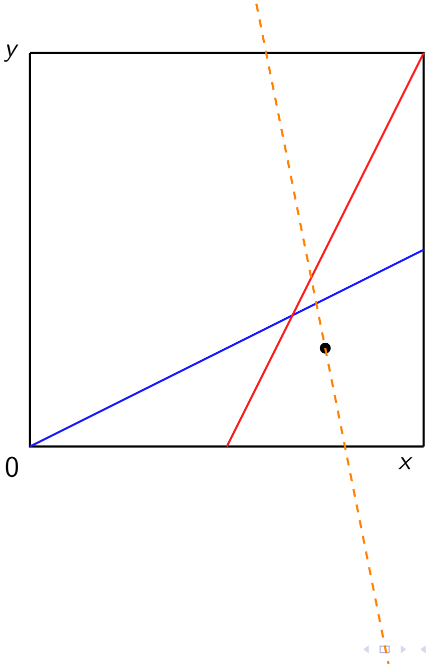

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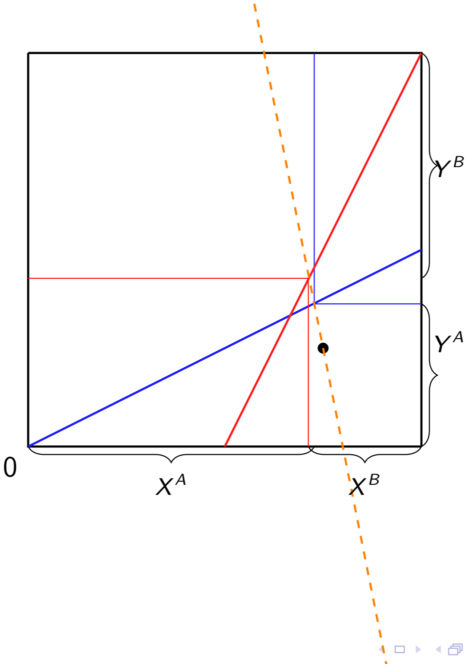
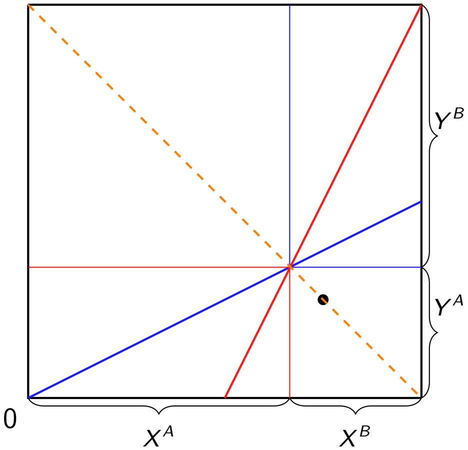


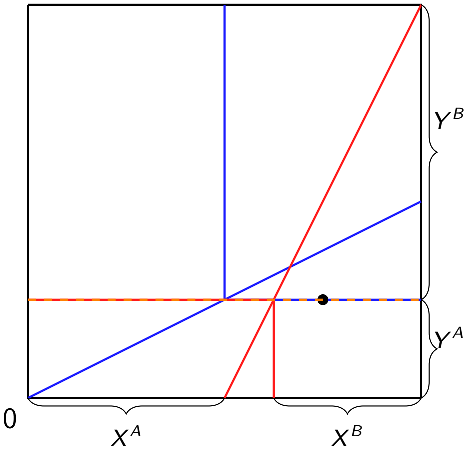
$\frac{p_x}{p_y} = 1$



No excess demand or supply

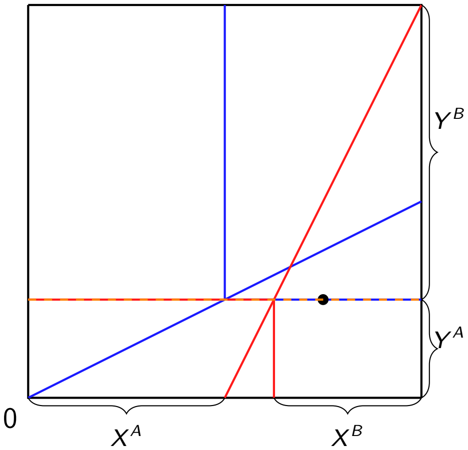
What about zero prices?

$p_x = 0$



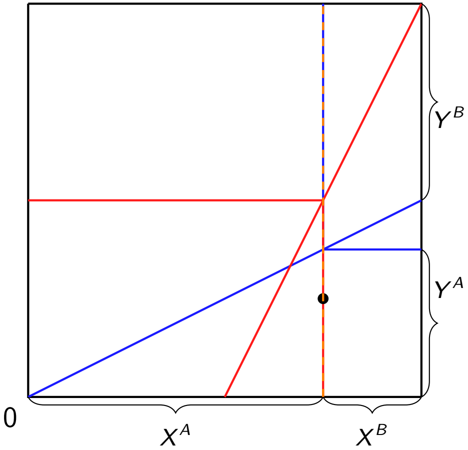
Excess supply of X? (and Y balanced?)

$p_x = 0$



Excess supply of X? (and Y balanced?) Not really since both A and B are indifferent over a wide range that would make the market clear

$p_y = 0$



Excess supply of Y? (and X balanced?)

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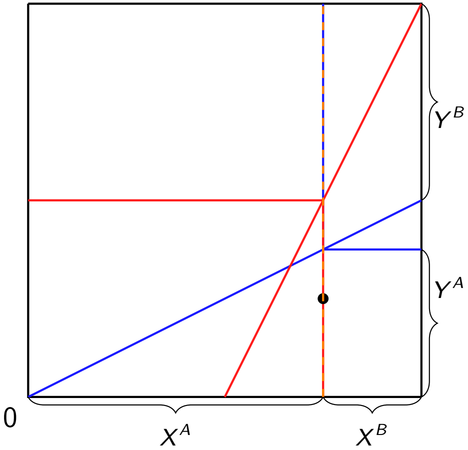

Excess supply of Y? (and X balanced?) Not really since both A and B are indifferent over a wide range that would make the market clear

To sum up...

- There are multiple equilibria
- There are three price vectors associated with these equilibria
- One price vector has a unique resource allocation associated with it
- Two price vectors ($p_x = 0$ and $p_y = 0$) have *infinity* resource allocations associated with them

Perfect complements

Try at home:

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, y^A) \\
u_B(x^B, y^B) &= \min(x^B, y^B) \\
\omega^A &= (1, 1) \\
\omega^B &= (3, 1)
\end{aligned}
$$

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Competitive equilibrium

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Perfect Substitutes

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$u_B(x^B, y^B) = x^B + y^B$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

Perfect Substitutes

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$u_B(x^B, y^B) = x^B + y^B$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

$p_x > 0$ and $p_y > 0$, why?

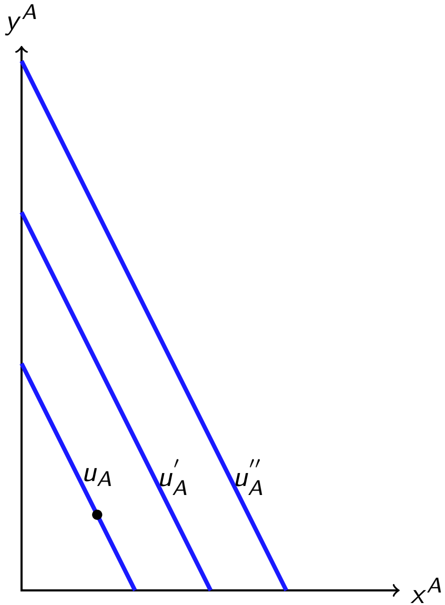
Perfect Substitutes

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$u_B(x^B, y^B) = x^B + y^B$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

$p_x > 0$ and $p_y > 0$, why? hence, normalize $p_x = 1$

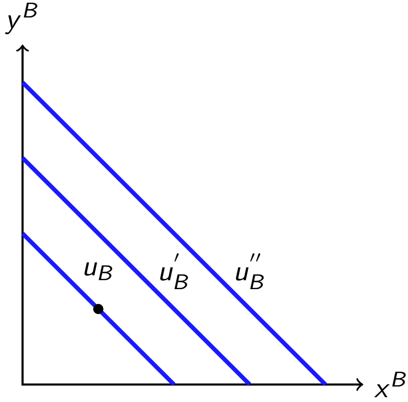
Perfect Substitutes

Peferences of person A:



Perfect Substitutes

Peferences of person B:



Algebraic solution

$$\max_{x^A, y^A} 2x^A + y^A$$

subject to:

$$I = x^A + p_y y^A$$
$$y^A \geq 0$$
$$x^A \geq 0$$

Perfect Substitutes

Algebraic solution

$$\max_{x^A, y^A} 2x^A + y^A$$

subject to:

$$I = x^A + p_y y^A$$
$$y^A \geq 0$$
$$x^A \geq 0$$

From the budget constraint we can obtain $y^A = \frac{I - x^A}{p_y}$, and adding the condition $y^A \geq 0$, we can conclude that $x^A \in [0, I]$.

Perfect Substitutes

Introducing y^A into the original maximization problem:

$$\max(2-\frac{1}{p_y})x^A+\frac{I}{p_y} \quad s.t.\ x^A \in [0, I]$$

Which is a maximization of a straight line with slope $\left(2-\frac{1}{p_y}\right)$ over an interval.

Perfect Substitutes

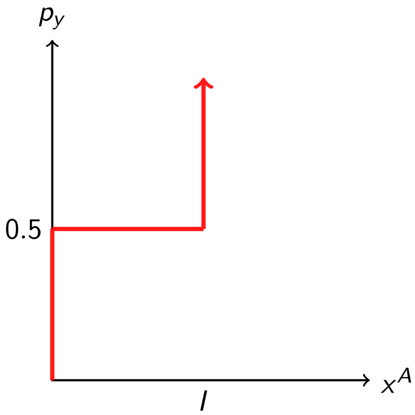
The demand for goods of individual A is

$$X^A = \begin{cases} 0 & \text{if } p_y < 0.5 \\ [0, I] & \text{if } p_y = 0.5 \\ I & \text{if } p_y > 0.5 \end{cases}$$

$$Y^A = \begin{cases} \frac{I}{p_y} & \text{if } p_y < 0.5 \\ [0, \frac{I}{p_y}] & \text{if } p_y = 0.5 \\ 0 & \text{if } p_y > 0.5 \end{cases}$$

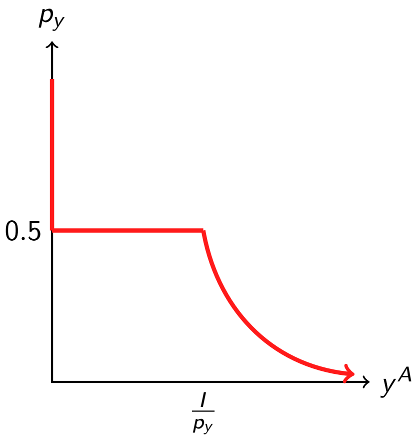
Perfect Substitutes

The demand for x^A is represented below:



Perfect Substitutes

The demand for y^A is represented below:



Algebraic solution

For person B the solution is analogous, but we have the following maximization problem: Introducing y^A into the original maximization problem:

$$\max(1 - \frac{1}{p_y})x^B + \frac{I}{p_y} \quad s.t.\, x^B \in [0, I]$$

Which is a maximization of a straight line with slope $\left(1 - \frac{1}{p_y}\right)$ over an interval.

Perfect Substitutes

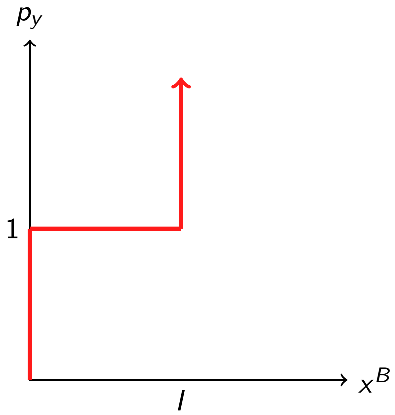
The demand for goods of individual B is

$$X^B = \begin{cases} 0 & \text{if } p_y < 1 \\ [0, I] & \text{if } p_y = 1 \\ I & \text{if } p_y > 1 \end{cases}$$

$$Y^B = \begin{cases} \frac{I}{p_y} & \text{if } p_y < 1 \\ [0, \frac{I}{p_y}] & \text{if } p_y = 1 \\ 0 & \text{if } p_y > 1 \end{cases}$$

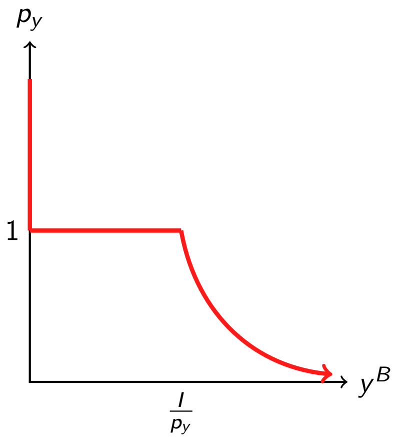
Perfect Substitutes

The demand for x^B is represented below:



Perfect Substitutes

The demand for y^B is represented below:



Perfect Substitutes

When is the market for good X balanced (how about good y?)

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- Try $p_y < 0.5$

Perfect Substitutes

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- $X^A = 0$ and $X^B = 0$

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When is the market for good X balanced (how about good y?)

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- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$
- $X^A = I$ and $X^B = 0$

Perfect Substitutes

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- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$
- $X^A = I$ and $X^B = 0$
- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$
- $X^A = I$ and $X^B = 0$
- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$
- $X^A = I = 2$ and $X^B = [0, 2]$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
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- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$
- $X^A = I = 2$ and $X^B = [0, 2]$
- One possible equilibrium $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
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- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$
- $X^A = I = 2$ and $X^B = [0, 2]$
- One possible equilibrium $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$
- Try $p_y > 1$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$
- $X^A = I$ and $X^B = 0$
- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$
- $X^A = I = 2$ and $X^B = [0, 2]$
- One possible equilibrium $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$
- Try $p_y > 1$
- $X^A = I$ and $X^B = I$

Perfect Substitutes

When is the market for good X balanced (how about good y?)

- Try $p_y < 0.5$
- $X^A = 0$ and $X^B = 0$
- Try $p_y = 0.5$
- $X^A = [0, I]$ and $X^B = 0$
- Can't be an equilibrium since $I = 1.5$ when $p_y = 0.5$, thus $X^A + X^B < 2$
- Try $0.5 < p_y < 1$
- $X^A = I$ and $X^B = 0$
- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B < 2$
- Try $p_y = 1$
- $X^A = I = 2$ and $X^B = [0, 2]$
- One possible equilibrium $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$
- Try $p_y > 1$
- $X^A = I$ and $X^B = I$
- $X^A = I = 1 + p_y$ and $X^B = I = 1 + p_y$
- Can't be an equilibrium since $I = 1 + p_y$, thus $X^A + X^B = 2 + 2p_y > 2$