Lecture 5: General Equilibrium

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What about the producers in the economy? There are two cases.

1. There are no producers in the economy: this is what is called a pure exchange economy in which all available goods are those coming from endowments from consumers (up until now)
2. There are producers who can produce commodities in the economy (today)

Each firm j is characterized by two characteristics:

1. A production function f_j^l for producing that good l.

The firm j has a production function of the form:

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- $z^{j,l}$ for firm j describes the vector of inputs that firm j uses in the production of good $\ell(l)$
- In other words, firm j uses $z_i^{j,l}$ units of commodity i to produce commodity l

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- An implicit assumption here is that firms do not have any endowments

Lecture 5: General Equilibrium

Introducing production

General equilibrium with production

Examples

Lecture 5: General Equilibrium

Definition

$((x^*, z^*), p = (p_1, \ldots, p_L))$ is a competitive equilibrium if:

1. For all producers $j = 1, 2, \ldots, J$,
   $z^{j^*} = ((z_1^{j,1^*}, \ldots, z_L^{j,1^*}), \ldots, (z_1^{j,L^*}, \ldots, z_L^{j,L^*}))$ solves:

$$\Pi_j^* := \max_{z^j} p_\ell f_j^\ell(z_1^j, \ldots, z_L^j) - \sum_{\ell'=1}^{L} p_{\ell'} z_{\ell'}^j.$$

2. For all consumers $i = 1, 2, \ldots, I$, $x^{i^*} = (x_1^{i^*}, \ldots, x_L^{i^*})$ solves:

$$\max_{x^i} u_i(x^i)$$

$$\text{such that } p \cdot x^i \leq p \cdot \omega^i + \sum_{j=1}^{J} \theta_{ij} \Pi_j^*.$$

3. Markets clear: For each commodity $\ell = 1, 2, \ldots, L$:

$$\sum_{i=1}^{I} x_\ell^{i^*} + \sum_{j=1}^{J} \sum_{\ell'=1}^{L} z_\ell^{j,\ell'} = \sum_{i=1}^{I} \omega_\ell^i + \sum_{j=1}^{J} f_j^\ell \left(z_1^{j,\ell^*}, \ldots, z_L^{j,\ell^*} \right).$$

We have exactly the same basic properties as in the case of pure exchange economies

1. When utility functions are strictly monotone, and production functions are strictly increasing, prices of each commodity and prices of each input are strictly positive
2. Walras' Law: Each consumer i spends all of his income whenever i maximizes utility
3. Walra's Law II: If the market clearing conditions hold for $\ell = 1, 2, \ldots, L-1$ and $p_L > 0$ then it will also hold for market L as well.
4. If (x^*, z^*, p) is a Walrasian equilibrium, and $\alpha > 0$, $(x^*, z^*, \alpha p)$ is also a Walrasian equilbrium.
5. The first and the second welfare theorems hold

Lecture 5: General Equilibrium

1. Imagine the problem of Robinson Crusoe, living alone in an island. He is the only producer and the only consumer

Suppose that the consumer (Robinson) has a utility function:

$$u(x, L) = x^{\alpha} L^{1-\alpha},$$

where x are coconuts. There is one firm (Robinson) that can convert labor to coconuts:

$$f_x(L) = L_x^{\beta}.$$

The endowment is $(0, \bar{L})$

A competitive equilibrium (x^*, L^*, L_x^*, p, w) satisfies the following:

1. L_x^* solves the following maximization problem:

$$\Pi^* := \max_{L_x} pL_x^\beta - wL_x.$$

2. (x^*, L^*) satisfies the following:

$$\max_{x,L} x^\alpha L^{1-\alpha} \text{ such that } px + wL \leq w\bar{L} + \Pi^*.$$

3. $x^* = {L_x^*}^\beta$ and $L + L_x = \bar{L}$.

- $p_x = 0$ and $w = 0$ cannot happen in a competitive equilibrium (why?)

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- $p_x = 0$ and $w > 0$ cannot happen (why?)
- both $p_x, w > 0$ in a competitive equilibrium. We can normalize $w = 1$

The problem of the firm

- We first solve the profit maximization
- This is usually a good first step because the profit enters into the demand function
- We first solve the profit maximization

For any $(p, w = 1)$, we want to solve:

$$\max_{L_x} pL_x^{\beta} - L_x.$$

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Therefore,

$$\Pi^*(p) = p\,(p\beta)^{\frac{\beta}{1-\beta}} - (p\beta)^{\frac{1}{1-\beta}} = p^{\frac{1}{1-\beta}}\left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}\right).$$

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The supply of x is then given by:

$$x^s(p) = L_x^*(p)^{\beta} = (p\beta)^{\frac{\beta}{1-\beta}}$$

The problem of the consumer

To solve for the demand curve $x^d(p), L^d(p)$, we solve:

$$\max_{x,L} x^\alpha L^{1-\alpha} \text{ such that } px + L \le \bar{L} + \Pi^*(p).$$

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Solving this, we obtain:

$$p^* = \left(\frac{\alpha\bar{L}}{\alpha\beta^{\frac{1}{1-\beta}} + (1-\alpha)\beta^{\frac{\beta}{1-\beta}}}\right)^{1-\beta}.$$

To solve for x^*, L^*, L_x^*, we plug the price back into the demand and supply functions:

$$x^* = x^s(p^*) = \left(\frac{\beta^{\frac{1}{1-\beta}} \alpha \bar{L}}{\beta^{\frac{\beta}{1-\beta}}(1-\alpha) + \alpha \beta^{\frac{1}{1-\beta}}} \right)^{\beta}$$

$$L^* = \bar{L} - L_x^* = \frac{1-\alpha}{1-\alpha+\alpha\beta}\bar{L}$$

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We can also solve for the profits of the firm in equilibrium:

$$\Pi^*(p^*) = p^{*\,\frac{1}{1-\beta}} \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right)$$

$$= \frac{\alpha \bar{L}}{\alpha \beta^{\frac{1}{1-\beta}} + (1-\alpha)\beta^{\frac{\beta}{1-\beta}}} \left(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right).$$

What is the Pareto optimal allocation in this economy? Try it at home

Suppose that there is one consumer with a utility function:

$$u(x, y) = x^{1/2}y^{1/2}.$$

There are two firms:

$$f_x(L_x, K_x) = L_x^{1/2}K_x^{1/2},$$
$$f_y(L_y, K_y) = L_y^{1/2}K_y^{1/2}.$$

The endowments are given by $\bar{L} = 1$, $\bar{K} = 1$, and 0 units of x and y.

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$$(x^*, y^*, L_x^*, K_x^*, L_y^*, K_y^*, p_x, p_y, r, w).$$

All equilibrium prices will be strictly positive in equilibrium, hence assume $p_x = 1$

A competitive equilibrium must satisfy the following conditions:

1. Profit maximization problems: (L_x^*, K_x^*) solves:

$$\Pi_x^* := \max_{L_x, K_x} f_x(L_x, K_x) - wL_x - rK_x$$

   (L_y^*, K_y^*) solves:

$$\Pi_y^* := \max_{L_y, K_y} p_y f_y(L_y, K_y) - wL_y - rK_y$$

2. Utility maximization: (x^*, y^*) solves:

$$\max_{x,y} \sqrt{xy} \text{ such that } x + p_y y \leq r\bar{K} + w\bar{L} + \Pi_x^* + \Pi_y^*.$$

3. Markets clear:

$$x^* = f_x(L_x^*, K_x^*), y^* = f_y(L_y^*, K_y^*), L_x^* + L_y^* = \bar{L}, K_x^* + K_y^* = \bar{K}.$$

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- Both firms make zero profits. Why?
  - This does not always happen (In the previous example, the firm made strictly positive profits)
  - this is because the production function here is of *constant returns to scale*
  - If the firm made strictly positive profits, then it could not be making maximal profits since it could double profits by multiplying all inputs by two

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For any $(p_x = 1, p_y, w, r)$, we want to solve:

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$$\frac{K_x}{L_x} = \frac{w}{r}$$

Therefore,

$$\begin{aligned}
\left(\frac{w}{r}\right)^{1/2} L_x^{1/2} L_x^{1/2} - L_x w - \frac{w}{r} L_x r &= 0 \\
\left(\frac{w}{r}\right)^{1/2} - 2w &= 0 \\
\frac{1}{2} &= w^{1/2} r^{1/2} \\
\frac{1}{4} &= wr
\end{aligned}$$

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The problem of the firm

Therefore,

$$
\begin{aligned}
\frac{K_y}{L_y} &= \frac{K_x}{L_x} \\
\frac{1-K_x}{1-L_x} &= \frac{K_x}{L_x} \\
L_x - K_x L_x &= K_x - K_x L_x \\
L_x &= K_x \\
w &= r \\
\frac{1}{4} &= wr \\
w = 1, r &= 1
\end{aligned}
$$

We also know that $p_y = 1$. Why?

The problem of the firm

- We cannot solve for the supply function because the firm obtains zero profit regardless of how much it produces
- But we already know the prices!

The problem of the consumer

$$\max_{x,y} \sqrt{xy} \text{ such that } x + y \leq 1.$$

The solution to this gives:

$$x^* = y^* = \frac{1}{2}.$$

market clearing

By market clearing we must have:

$$\frac{1}{2} = L_x^* = K_x^* = L_y^* = K_y^*,$$