Lecture 12

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Lecture12

Lecture 12: Game Theory // Nash equilibrium

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Lecture 12: Game Theory // Nash equilibrium

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Beauty contest

- Consider the following game among 100 people. Each individual selects a number, s_i, between 20 and 60.
- Let a_{-i} be the average of the number selected by the other 99 people. i.e. $a_{-i} = \sum_{j \neq i} \frac{s_j}{99}$.
- The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - \underbrace{(s_i - \frac{3}{2}a_{-i})^2}_{+}$

$\rightarrow s_i - \frac{3}{2}a_{-i} = 0$

$\Leftrightarrow \boxed{s_i = \frac{3}{2}a_{-i}}$

$\frac{\partial u_i}{\partial s_i} = -2(s_i - \frac{3}{2}a_{-i}) = 0$

Beauty contest

- Each individual maximizes his utility, FOC:

$$-2(s_i - \frac{3}{2}a_{-i}) = 0$$

Beauty contest

- Each individual maximizes his utility, FOC:

$$-2(s_i - \frac{3}{2}a_{-i}) = 0 \rightarrow \boxed{s_i = \frac{3}{2}a_{-i}}$$

- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others

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- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$

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- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- but $\boxed{a_{-i} \in [20, 60]}$ $\rightarrow \frac{3}{2}a_{-i} \in [\frac{3}{2}20, \frac{3}{2}60] = [30, 90]$

Beauty contest

- Each individual maximizes his utility, FOC:

$$-2(s_i - \frac{3}{2}a_{-i}) = 0$$

- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- but $a_{-i} \in [20, 60]$
- Therefore $s_i = 20$ is dominated by $s_i = 30$

Beauty contest

- The same goes for any number between 20 (inclusive) and 30 (not included)

30>>21
30>>20
30>>22
⋮
30>>29

Beauty contest

- The same goes for any number between 20 (inclusive) and 30 (not included)
- Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$) $\rightarrow \frac{3}{2}a_{-i} \in [45, 90]$

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- The same goes for any number between 20 (inclusive) and 30 (not included)
- Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$)
- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45

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- The same goes for any number between 20 (inclusive) and 30 (not included)
- Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$)
- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45
- Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$) $\Rightarrow \frac{3}{2}a_{-i} \in [67.5, 90]$

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- Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$)
- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45
- Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$)
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- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45
- Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$)
- 60 would dominate any other selection and therefore all the players select 60.
- The solution by means of iterated elimination of dominated strategies is $\boxed{(\underbrace{60, 60, \ldots, 60}_{100 \text{ times}})}$

$$U_i(60, \ldots, 60) = 100 - \left(60 - \tfrac{3}{2} 60\right)^2 = -800$$

$$U_i(20, \ldots, 20) = 100 - \left(20 - \tfrac{3}{2} 20\right)^2 = 0$$

$\Rightarrow (20, \ldots, 20)$ Pareto domina $\boxed{(60, \ldots, 60)}$

↳ ¿Por qué no es EQ/solución o juego?

↳ "Yo tengo incentivos a desviarme a 30"

Lecture 12: Game Theory // Nash equilibrium

Dominance
- Weakly dominated strategies

Nash equilibrium

Some examples

Relationship to dominance

Examples
- Cournot Competition
- Cartels

	a	b
A	[illegible]	[illegible]
B	[illegible]	[illegible]
C	5, 3	4, 3

- $C \geq B \to a \geq b \to C > A \to (C, a)$
- $C \geq A \to b \geq a \to C > B \to (C, b)$
- $C \geq B \wedge C \geq A \to a \sim b \to$ No soluble

- There is no strictly dominated strategy

↳ Eliminación iterada de estrategias débilmente dominadas no es una técnica robusta

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- There is no strictly dominated strategy
- However, C always gives at least the same utility to player 1 as B

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- There is no strictly dominated strategy
- However, C always gives at least the same utility to player 1 as B
- It's tempting to think player 1 would never play C
- However, if player 1 is sure that player two is going to play a he would be completely indifferent between playing B or C

Definition

s_i weakly dominates s_i' if for all opponent pure strategy profiles, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \boxed{\geq} u_i(s_i', s_{-i})$$

and there is at least one opponent strategy profile $s''_{-i} \in S_{-i}$ for which

$$u_i(s_i, s''_{-i}) \boxed{>} u_i(s_i', s''_{-i}).$$

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- Given the assumptions we have, we can not eliminate a weakly dominated strategy
- Rationality is not enough
- Even so, it sounds "logical" to do so and has the potential to greatly simplify a game
- There is a problem, and that is that the order in which we eliminate the strategies matters

---

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).

---

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).
- If on the other hand, we notice that A is also weakly dominated by C then we can eliminate it in the first round, and this would eliminate a in the second round and therefore B would be eliminated. This would result in (C, b).

Lecture 12: Game Theory // Nash equilibrium

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Dominance

Nash equilibrium

Some examples

Relationship to dominance

Examples

Remember the definition of competitive equilibrium in a market economy.

Definition

A competitive equilibrium in a market economy is a vector of prices and baskets x_i such that: 1) x_i maximizes the utility of each individual given the price vector i.e.

$$x_i = \arg \max_{p\ cdotx_i \leq p \cdot w_i} u(x_i)$$

2) the markets empty.

$$\sum_i x_i = \sum_i w_i$$

- 1) means that given the prices, individuals have no incentive to demand a different amount

- 1) means that given the prices, individuals have no incentive to demand a different amount

- The idea is to extend this concept to strategic situations

Best response

$\to BR_i(S_{-i})$

We denote $BR_i(s_{-i})$ (best response) as the set of strategies of individual i that maximize her utility given that other individuals follow the strategy profile s_{-i}. Formally,

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Definition

Given a strategy profile of opponents s_{-i}, we can define the best response of player i:

$$BR_i(s_{-i}) = \arg \max_{s_i' \in S_i} u_i(s_i', s_{-i}).$$

- $s_i \in BR_i(\boxed{s_{-i}})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$

$$\boxed{\begin{array}{l} \max_x f(x) = f(x^*) \\ \text{ARG} \max_x f(x) = x^* \end{array}}$$

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$$BR_i(s_{-i}) = \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}).$$

- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$
- There could be multiple strategies in $BR_i(s_{-i})$ but all such strategies give the same utility to player i if the opponents are indeed playing according to s_{-i}

$s_i^* \in BR_i(s_{-i})$
$\hat{s}_i^* \in BR_i(s_{-i})$ $\Rightarrow$ $\boxed{u_i(s_i^*, s_{-i}) = u_i(\hat{s}_i^*, s_{-i})}$

Nash equilibrium

Definition
Suppose that we have a game $\underbrace{(I = \{1, 2, \ldots n\}, S_1, \ldots, S_n, u_1, \ldots, u_n)}_{\text{NORMAL}}$. Then a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a pure strategy **Nash equilibrium** if for every i and for every $s_i \in S_i$,

$$\boxed{u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).} \rightarrow s_i^* \in BR_i(s_{-i}^*)$$

$\rightarrow s_1^* \neq s_2^*$

$\boxed{s^*}$

Nash equilibrium

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Suppose that we have a game $(I = \{1, 2, \ldots, n\}, S_1, \ldots, S_n, u_1, \ldots, u_n)$. Then a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **pure strategy** Nash equilibrium if for every i, $s_i^* \in BR_i(s_{-i}^*)$.

- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate

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Suppose that we have a game $(I = \{1, 2, \ldots, n\}, S_1, \ldots, S_n, u_1, \ldots, u_n)$. Then a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **pure strategy** Nash equilibrium if for every i, $s_i^* \in BR_i(s_{-i}^*)$.

- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- once this equilibrium is reached, nobody has incentives to move from there

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Suppose that we have a game $(I = \{1, 2, \ldots, n\}, S_1, \ldots, S_n, u_1, \ldots, u_n)$. Then a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **pure strategy** Nash equilibrium if for every i, $s_i^* \in BR_i(s_{-i}^*)$.

- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- once this equilibrium is reached, nobody has incentives to move from there
- This is a concept of stability, but there is no way to ensure, or predict, that the game will reach this equilibrium

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Beauty contest

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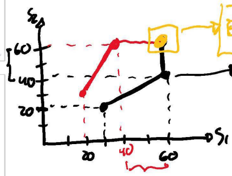
- Let s_{-i} be the number selected by the other individual.
- The utility function of the individual i is $\boxed{u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}\underbrace{s_{-i}}_{a-i})^2}$

$\boxed{U_1(s_1,s_2) = 100 - (s_1 - \frac{3}{2}s_2)^2}$

$U_2(s_1,s_2) = 100 - (s_2 - \frac{3}{2}s_1)^2$

$MR_1(s_2) \Rightarrow \frac{\partial U_1(s_1,s_2)}{\partial s_1} = -2(s_1 - \frac{3}{2}s_2) = 0$

$\Rightarrow \boxed{s_1^* = \frac{3}{2}s_2} \Rightarrow MR_1(s_2) = \begin{cases} \frac{3}{2}s_2 & \text{si } s_2 \leq 40 \\ 60 & \text{si } s_2 \geq 40 \end{cases}$



$EN = (60,60)$

$MR_2(s_1) = \begin{cases} \frac{3}{2}s_1 & \text{si } s_1 \leq 40 \\ 60 & \text{si } s_1 \geq 40 \end{cases}$

Beauty contest

The best response of an individual is given by

$$s_i(s_{-i})^* = \begin{cases} \frac{3}{2}s_{-i} & \text{if } s_{-i} \leq 40 \\ 60 & \text{if } s_{-i} > 40 \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)

$U_1(20,20) > U_1(60,60)$

$MR_1(20) = 30$

$U_1(30,20) > U_1(20,20) \rightarrow$ S_1 TIENE INCENTIVOS UNILATERALES A DESVIARSE DE $(20,20)$

$\boxed{U_1(60,60) \geq U_1(x,60)}$ $x \in [20,60]$

Prisoner's dilemma

s_2

s_1	C	NC
C	5,5	0,10
NC	10,0	2,2

$MR_1(s_2 = C) = NC$

$MR_1(s_2 = NC) = NC$

$MR_1(s_2) = NC$

$MR_2(s_1) = NC$

$EN = (NC,NC)$

Prisoner's dilemma

	C	NC
C	5,5	0,10
NC	10,0	2,2

The best response functions are:

$$BR_i(s_{-i}) = \begin{cases} NC & \text{if } s_{-i} = C \\ NC & \text{if } s_{-i} = NC \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play NC, i.e., (NC, NC))

Prisoner's dilemma – A trick

Best response of 1 to 2 playing C

s_2

	C	NC
C	5,5	0,10
NC	10,0	2,2

$EN = (NC,NC)$

(NC,NC) no es O.P.

Prisoner's dilemma – A trick

Best response of 1 to 2 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing C

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

When underlined for both players, it is a Nash equilibrium (both are doing their BR)

Battle of the sexes

s_2

s_1

	G	P
G	2,1	0,0
P	0,0	1,2

EN = {(G,G); (P,P)}.

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Thus, (G,G) y (P,P) are both Nash equilibrium

Matching pennies (Pares o Nones) – Simultaneous

s_2

s_1

	1	2
1	(1000,-1000)	(-1000,1000)
2	(-1000,1000)	(1000,-1000)

$(1,2) \downarrow (1,1) \downarrow (1,2)$

Matching pennies (Pares o Nones) – Simultaneous

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Matching pennies (Pares o Nones) – Simultaneous

	1	2
1	(1000,-1000)	(-1000,1000)
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$$BR_1(s_2) = \begin{cases} 1 & \text{if } s_2 = 1 \\ 2 & \text{if } s_2 = 2 \end{cases}$$

$$BR_2(s_1) = \begin{cases} 2 & \text{if } s_1 = 1 \\ 1 & \text{if } s_2 = 2 \end{cases}$$

There is no Nash equilibrium in pure strategies

Lecture 12: Game Theory // Nash equilibrium

Dominance

Nash equilibrium

Some examples

Relationship to dominance

Examples

Lecture 12: Game Theory // Nash equilibrium

Dominance

Nash equilibrium ✓

Some examples ✓

Relationship to dominance

Examples

- Solubles eliminacion iterada.. = E. N.
- Algunos no solubles eliminacion iterada... si tienen E. N.
- Hay algunos juegos sin solución.

Nash equilibrium survive IDSDS

Theorem
Every Nash equilibrium survives the iterative elimination of strictly dominated strategies

no es eliminado

Proof

By contradiction:

- Suppose it is not true

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- Lets zoom in in the round where we first eliminate a strategy that is part of s^*

Proof

By contradiction:

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- Then we must have eliminated some strategy in the Nash equilibrium $s^* = (s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*)$
- Lets zoom in in the round where we first eliminate a strategy that is part of s^*
- Without loss of generality say we eliminated the strategy s_i^* of individual i

Proof

By contradiction:

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- Then we must have eliminated some strategy in the Nash equilibrium s^*
- Lets zoom in in the round where we first eliminate a strategy that is part of s^*
- Without loss of generality say we eliminated the strategy s_i^* of individual i
- It must have been that

$$u_i(s_i^*, s_{-i}) < u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

Si Domina S_i^*

En particular s_{-i}^*

.

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- In particular

$$\boxed{u_i(s_i^*, s_{-i}*) < u_i(s_i, s_{-i}^*)}$$

$\to s_i^* \notin MR_i(s_{-i}^*)$

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By contradiction:

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$$u_i(s_i^*, s_{-i}*) < u_i(s_i, s_{-i}^*)$$

- But this means s_i^* is not the best response of individual i to s_{-i}^*

Proof

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$$u_i(s_i^*, s_{-i}*) < u_i(s_i, s_{-i}^*)$$

- But this means s_i^* is not the best response of individual i to s_{-i}^*
- And this is a contradiction!

Nash equilibrium survive IDSDS

Theorem

If the process of IDSDS comes to a single solution, that solution is a Nash Equilibrium and is unique.

Soluble por IDSDS

Por teorema anterior

	G	P
G	2,1	0,0
P	0,0	1,2

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium

□

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

$$\boxed{u_i(s_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i})}$$

□

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By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

$$u_i(s_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i})$$

- But then s_i could not have been eliminated

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By contradiction:

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□

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Cournot Competition

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- Suppose that there are two firms that produce the same product have zero marginal cost of production.
- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

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- Strategy space is $S_i = [0, +\infty)$

Cournot Competition → FIRMAS COMPITEN EN CANTIDADES.

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- Suppose that there are two firms that produce the same product have zero marginal cost of production.
- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$\boxed{P(Q) = 120 - Q}, Q = q_1 + q_2.$$

- Strategy space is $S_i = [0, +\infty)$ → S_i → CUANTO PRODUCIR $= q_i$
- The utility function of player i is given by:

$$\boxed{\begin{aligned}\pi_1(q_1, q_2) &= (120 - (q_1 + q_2))q_1,\\ \pi_2(q_1, q_2) &= (120 - (q_1 + q_2))q_2.\end{aligned}}$$

$$\to \pi_1 = \underbrace{P(Q)}_{(120-q_1-q_2)} q_1 - \cancel{CT}(q_1)^{\,0}$$

$$\to \frac{\partial \pi_1}{\partial q_1} \Rightarrow 120 - 2q_1 - q_2 = 0$$

$$\boxed{\frac{120 - q_2}{2} = q_1^*}$$

Cournot Competition

- Are there any strictly dominant strategies?

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- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0

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- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0 $\rightarrow q_i \in [0, 120]$
- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

$$\boxed{q_i^* = \frac{120 - q_{-i}}{2}}$$

- Therefore 60 strictly dominates any $q_i \in (60, 120]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$

Cournot Competition

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- Such a q_i can never be strictly dominated

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- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$
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- After one round of deletion of strictly dominated strategies, we are left with: $S_i = [0, 60]$

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Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
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- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $\boxed{q_{-i} = [0, 60]}$ $q_i \in [0,60]$
- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$
- After two rounds of deletion of strictly dominated strategies, we are left with: $\boxed{S_i = [30, 60]}$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 60]$
- 45 strictly dominates all strategies $q_i \in (45, 60]$
- After three rounds of deletion of strictly dominated strategies, we are left with: $\boxed{S_i = [30, 45]}$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 45]$
- 37.5 strictly dominates all strategies $q_i \in [30, 37.5]$
- After four rounds of deletion of strictly dominated strategies, we are left with: $S_i = [37.5, 45]$

Cournot Competition

- After (infinitely) many iterations, the only remaining strategies are $\boxed{S_i = 40}$
- The unique solution by IDSDS is $q_1^* = q_2^* = 40$.

Cournot Competition

- There will also be a unique Nash equilibrium

Cournot Competition

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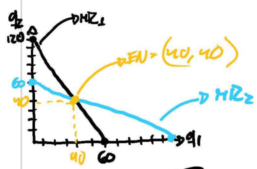
$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

$$MR_1(q_2) = \frac{120 - q_2}{2} = q_1^* \quad (1)$$

$$MR_2(q_1) = \frac{120 - q_1}{2} = q_2^* \quad (2)$$

→ 2 ECUACIONES
2 INCOGNITAS ⇒ SOLUCIÓN

EN = (40, 40)



Cournot Competition

- There will also be a unique Nash equilibrium
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$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- At any Nash equilibrium, we must have: $q_1^* \in BR_1(q_2^*)$ and $q_2^* \in BR_2(q_1^*)$.

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-
$$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$
- We can solve for q_1^* and q_2^* to obtain:

EN.

$$\boxed{q_1^* = 40, q_2^* = 40, Q^* = 80, \Pi_1^* = \Pi_2^* = 1600.}$$

Cournot Competition vs Monopoly (cartel)

$\boxed{P=0}$

- In a perfectly competitive market, price equals marginal cost and the total quantity produced will be $Q = 120$.

Cournot Competition vs $\boxed{\text{Monopoly (cartel)}}$

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- A monopolist would solve the following maximization problem:

$$\boxed{\max_{Q}(120 - Q)Q} \Rightarrow \boxed{Q^{*} = 60, P^{*} = 60, \Pi^{m} = 3600.}$$

$\max_{Q} P(Q)Q$

$\frac{\partial \Pi}{\partial Q} = 120 - 2Q = 0$

$\boxed{Q = 60}$

$q_1 = 30$

$q_2 = 30$

$Q = 60 = Q^M$

$\frac{\Pi^M}{2} = \Pi_1 = \Pi_2 = 1800$

$MR_i(30) = \frac{120-30}{2} = 45$

$\Pi_1(45, 30) > \Pi^M/2$

$\Pi_2(45, 30) < \frac{\Pi^M}{2}$

Cournot Competition vs Monopoly (cartel)

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- The profits to each firm in the Cournot Competition is less than half of the monopoly profits
- In a duopoly, externalities are imposed on the other firm

Cournot Competition - General case

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- To simplify notation, let $Q_{-i} = \sum_{j \neq i} q_j$
- $$\max_{q_i} p(q_i + Q_{-i})q_i - c_i(q_i)$$

Cournot Competition - General case

$$\max_{q_i} p(q_i + Q_{-i})q_i - c_i(q_i)$$

- First order condition implies:

$$\begin{aligned}
q_i \frac{dP}{dQ}(q_i + Q_{-i}) + P(q_i + Q_{-i}) &= \frac{dc_i}{dq_i}(q_i) \\
q_i \frac{dP}{dQ}(Q) + P(Q) &= \frac{dc_i}{dq_i}(q_i) \\
P(Q) - \frac{dc_i}{dq_i}(q_i) &= -q_i \frac{dP}{dQ}(Q) \\
\frac{P(Q) - \frac{dc_i}{dq_i}(q_i)}{P(Q)} &= -\frac{q_i}{Q} \frac{Q}{P(Q)} \frac{dP}{dQ}(Q) \\
\frac{P(Q) - \frac{dc_i}{dq_i}(q_i)}{P(Q)} &= -\frac{q_i}{Q} \frac{1}{\varepsilon_{Q,P}(Q)}
\end{aligned}$$

Cournot Competition - General case

$$\frac{P(Q) - \frac{dc_i}{dq_i}(q_i)}{P(Q)} = -\frac{q_i}{Q} \frac{1}{\varepsilon_{Q,P}(Q)}$$

- Therefore in a pure strategy Nash equilibrium $(q_1^*, q_2^*, \ldots, q_n^*)$ with $Q^* = q_1^* + q_2^* + \cdots q_n^*$, we must have:

$$\begin{aligned}
\frac{P(Q^*) - \frac{dc_1}{dq_1}(q_1^*)}{P(Q^*)} &= -\frac{q_1^*}{Q^*} \frac{1}{\varepsilon_{Q,P}(Q^*)}, \\
\frac{P(Q^*) - \frac{dc_2}{dq_2}(q_2^*)}{P(Q^*)} &= -\frac{q_2^*}{Q^*} \frac{1}{\varepsilon_{Q,P}(Q^*)}, \\
&\vdots \\
\frac{P(Q^*) - \frac{dc_n}{dq_n}(q_n^*)}{P(Q^*)} &= -\frac{q_n^*}{Q^*} \frac{1}{\varepsilon_{Q,P}(Q^*)}.
\end{aligned}$$

Cournot Competition - General case

- Suppose that all firms have exactly the same cost function c

$$\begin{aligned}
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- In this case $Q^* = nq^*$

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- Rewriting

$$P(Q^*) = \frac{1}{1 + \frac{1}{n} \frac{1}{\varepsilon_{Q,P}(Q^*)}} \frac{\partial c}{dq} \left(\frac{Q^*}{n} \right).$$

Lecture 12: Game Theory // Nash equilibrium

Cartels

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- The inverse demand function is given by:

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- In a Nash equilibrium we must have:

$$\begin{aligned} q_1^* &= \frac{1 - q_2^* - q_3^*}{2} \\ q_2^* &= \frac{1 - q_1^* - q_3^*}{2} \\ q_3^* &= \frac{1 - q_1^* - q_2^*}{2}. \end{aligned}$$

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- $q_1^* = q_2^* = q_3^* = \frac{1}{4}$
- Price is $p^* = 1/4$ and all firms get the same profits of $1/16$

Cartels

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- Therefore

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$$q_B^* = \frac{1 - q_A^*}{2}.$$

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- Firms 1 and 2 suffered, while firm 3 is better off!
- Firm 3 is obtaining a disproportionate share of the joint profits (more than $1/3$)

Cartels

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- Firm 3 clearly wants to stay out

Cartels

There are many ifficulties associated with sustaining collusive agreements (e.g., the OPEC cartel)