Lecture 13

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Lecture13

Lecture 13: Game Theory // Nash equilibrium

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Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

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Examples - Continued

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Examples - Continued
- Cournot - Revisited
- Bertrand Competition
- Bertrand Competition - Different costs
- Bertrand Competition - 3 Firms
- Hotelling and Voting Models

Cournot Competition

- N identical firms competing on the same market

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Cournot Competition

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- Marginal cost is constant and equal to c
- Aggregate inverse demand is

$$Q = \sum_{j=1}^{N} q^j$$

$$\boxed{p = a - b\sum_{j=1}^{N} q^j} = P(Q) = a - bQ$$

- Benefits of firm j are:

$$\Pi^j(q^1, \ldots q^N) = \underbrace{\left(a - b\sum_{i=1}^{N} q^i\right)}_{P(Q)} q^j - \underbrace{cq^j}_{C(q)} \longrightarrow \pi^j$$

NO DEBEN HACER

$$\boxed{\begin{aligned} &\Pi = \left(a - b\sum_{i=1}^{N} q_i\right) q_j - c q_j \\ &\cancel{\Pi = \left(a - b\sum q^*\right) q^* - c q^*} \\ &\cancel{\Pi = (a - bNq^*) q^* - c q^*} \\ &\cancel{\frac{\partial \Pi}{\partial q^*} = 0} \end{aligned}}$$

$$-bq_j$$

$$(a - bq_1 - bq_2 - \cdots - bq_N)\, q_j - c q_j$$

$$\boxed{\frac{\partial \pi^j}{\partial q_j}} = a - bq_1 - bq_2 - \cdots - bq_{j-1} - 2bq_j - bq_{j+1} \cdots - bq_N - c = 0$$

$$= \boxed{a - b\sum_{i=1}^{N} q_i - bq_j - c = 0}$$

$$\boxed{\begin{aligned} &\frac{\partial \Pi_1}{\partial q_1} = 0 \\ &\frac{\partial \Pi_2}{\partial q_2} = 0 \\ &\vdots \\ &\frac{\partial \Pi_N}{\partial q_N} = 0 \end{aligned}}$$

N ECUACIONES
N INCOGNITAS

⇒ Buscamos un EQ simétrico

$$\boxed{q_1^* = q_2^* = \cdots = q_N^* = q^*}$$

$$a - b\sum_{i=1}^{N} q^* - bq^* - c = 0$$

$$a - bNq^* - bq^* - c = 0$$

$$a - c = b(N+1)q^*$$

$$\boxed{\frac{a-c}{b(N+1)} = q^*}$$

Cournot Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

Cournot Competition

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$$q^* = \frac{a-c}{b(N+1)}$$

Cournot Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

- The symmetric Nash equilibrium is given by

$$q^* = \frac{a-c}{b(N+1)}$$

- Thus

$$\boxed{Q = \sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}}$$

$$p = a - N\frac{a-c}{(N+1)} < a \quad P = a - bQ$$

$$\boxed{\Pi^j = \frac{(a-c)^2}{b(N+1)^2}}$$

$$\boxed{\lim_{N\to\infty} P = \lim_{N\to\infty} a - \frac{N(a-c)}{N+1} = a - (a-c) = c = \text{Cmg}}$$

$$\boxed{\lim_{N\to\infty} \pi^j = \lim_{N\to\infty} \frac{(a-c)^2}{b(N+1)^2} = 0}$$

Cournot Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)} = Q$$

$$p = a - N\frac{a-c}{(N+1)} < a$$

$$\Pi^j = \frac{(a-c)^2}{b(N+1)^2}$$

- As $N \to \infty$ we get close to perfect competition

Cournot Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$

$$p = a - N\frac{a-c}{(N+1)} < a$$

$$\Pi^j = \frac{(a-c)^2}{b(N+1)^2}$$

- As $N \to \infty$ we get close to perfect competition
- $\boxed{N = 1 \text{ we get the monopoly case}}$

$N=1$

$Q^{EN} = q^M$

$P^{EN} = P^M$

$\pi^{EN} = \pi^M$

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

Cournot - Revisited

Bertrand Competition

Bertrand Competition - Different costs

Bertrand Competition - 3 Firms

Hotelling and Voting Models

Bertrand Competition

- Consider the alternative model in which firms set prices
- In the monopolist's problem, there was not distinction between a quantity-setting model and a price setting
- In oligopolistic models, this distinction is very important

Bertrand Competition

- Consider two firms with the same marginal constant marginal cost of production and demand is completely inelastic
- Each firm simultaneously chooses a price $p_i \in [0, +\infty)$
- If p_1, p_2 are the chosen prices, then the utility functions of firm i is given by:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 0 & \text{if } \boxed{p_i > p_{-i}} \\ (p_i - c)\frac{Q(p_i)}{2} & \text{if } p_i = p_{-i}, \\ (p_i - c)Q(p_i) & \text{if } \boxed{p_i < p_{-i}} \end{cases}$$

Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$R(p_i) = p_i Q(p_i) \quad (1)$$

$$MR(p_i) = Q(p_i) + p_i Q'(p_i) \quad (2)$$

$$= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). \quad (3)$$

Bertrand Competition

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- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

Bertrand Competition

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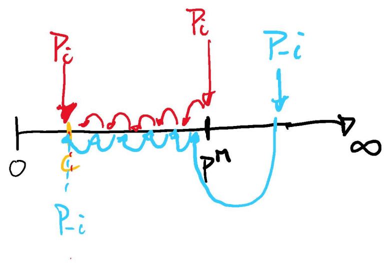
$$
\begin{aligned}
R(p_i) &= p_i Q(p_i) && (1)\\
MR(p_i) &= Q(p_i) + p_i Q'(p_i) && (2)\\
&= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). && (3)
\end{aligned}
$$

- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.
- Then

$$MR(p_i) - c > 0 \text{ if } p_i < p^m, MR(p_i) - c < 0 \text{ if } p_i > p^m.$$



Bertrand Competition

- The best response function is:

$$BR_i(p_{-i}) = \begin{cases} p^m & \text{if } p_{-i} > p^m, \\ p_{-i} - \varepsilon & \text{if } c < p_{-i} \leq p^m, \\ [c, +\infty) & \text{if } c = p_{-i} \\ (c, +\infty) & \text{if } c > p_{-i}. \end{cases}$$

- Where ε is the smallest monetary unit

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$

Bertrand Competition

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- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$

Bertrand Competition

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Bertrand Competition

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- So this cannot be a Nash equilibrium

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

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Bertrand Competition

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Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$

Bertrand Competition

Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$
- The unique pure strategy Nash equilibrium is $p_1^* = p_2^* = c$

Bertrand Competition

Thus in contrast to the Cournot duopoly model, in the Bertrand competition model, two firms get us back to perfect competition ($p = c$)

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued
- Cournot - Revisited
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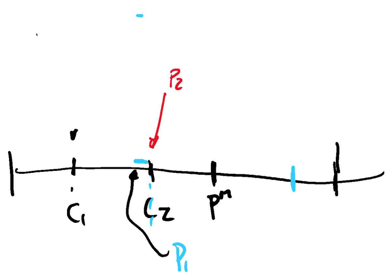
- Bertrand Competition - Different costs
- Bertrand Competition - 3 Firms
- Hotelling and Voting Models

Bertrand Competition - different costs

- Suppose that the marginal cost of firm 1 is equal to c_1 and the marginal cost of firm 2 is equal to c_2 where $c_1 < c_2$.
- The best response for each firm:

$$BR_i(p_{-i}) = \begin{cases} p_m^i & \text{if } p_{-i} > p_m^i, \\ p_{-i} - \varepsilon & \text{if } c_i < p_{-i} \le p_m^i, \\ [c_i, +\infty) & \text{if } p_{-i} = c_i \\ (p_{-i}, +\infty) & \text{if } p_{-i} < c_i. \end{cases}$$



Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit

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- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices

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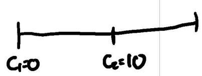
- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices
- No NE because of continuous prices

Bertrand Competition - discreet prices



- Suppose $c_1 = 0 < c_2 = 10$

Bertrand Competition - discreet prices

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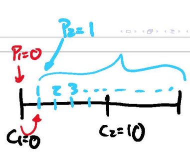
- Firms can only set integer prices.

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.
- Suppose that (p_1^*, p_2^*) is a pure strategy Nash equilibrium...

Bertrand Competition - discreet prices



Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$

Bertrand Competition - discreet prices

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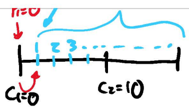


- Best response of firm 2 is to choose some $p_2^* > p_1^*$

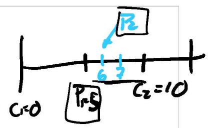
Bertrand Competition - discreet prices

Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$
- p_1^* cannot be a best response to p_2^* since by setting $p_1 = p_2^*$ firm 1 would get strictly positive profits

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$



- Best response of firm 2 is to set any price $p_2^* > p_1^*$

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

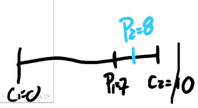
- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price

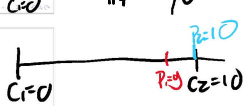
Bertrand Competition - discreet prices

Case 2: $\boxed{p_1^* \in \{1, 2, \ldots, 9\}}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price
- The only equilibrium is $(p_1^*, p_1^* + 1)$

$$EN = \left\{ \begin{array}{l} (P_1, P_2) \\ (1, 2) \\ (2, 3) \\ (3, 4) \\ (4, 5) \\ (5, 6) \\ (6, 7) \\ (7, 8) \\ (8, 9) \\ (9, 10) \end{array} \right.$$

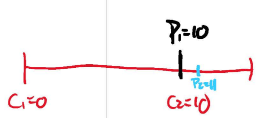




Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$



Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

Bertrand Competition - discreet prices

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- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
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$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher

Bertrand Competition - discreet prices

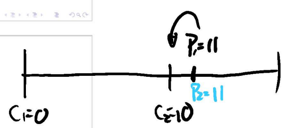
Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher
- $(p_1^*, p_2^*) = (10, 11)$ is a Nash equilibrium

Bertrand Competition - discreet prices



Case 4: $p_1^* = 11$

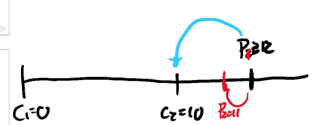
- Best response of firm 2 is to set $p_2^* = 11$

Bertrand Competition - discreet prices

Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$
- Firm 1 would not be best responding since by setting a price of $p_1 = 10$, it would get strictly positive profits

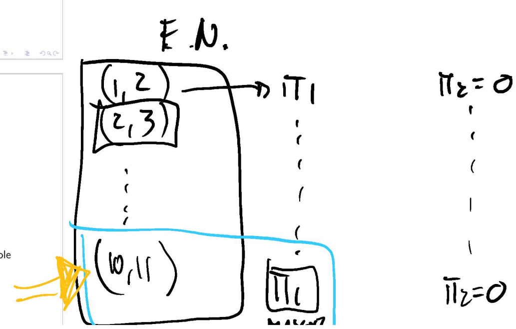
Bertrand Competition - discreet prices



Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$

Bertrand Competition - discreet prices



Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$
- Firm 1 is not best responding since by lowering the price it can get the whole market.

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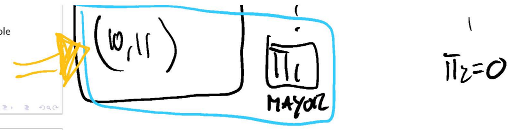

Lecture 13: Game Theory // Nash equilibrium

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Hotelling and Voting Models

Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms

Bertrand Competition - 3 firms

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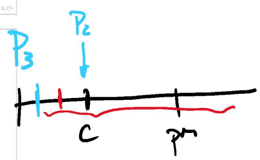
- Best response of firm i is given by:

$$BR_1(p_2,p_3)=\begin{cases} p^m & \text{if } \min\{p_2,p_3\}>p^m,\\ \min\{p_2,p_3\}-\varepsilon & \text{if } c<\min\{p_2,p_3\}\le p^m,\\ [c,+\infty) & \text{if } c=\min\{p_2,p_3\},\\ (\min\{p_2,p_3\},+\infty) & \text{if } c>\min\{p_2,p_3\}.\end{cases}$$



Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms
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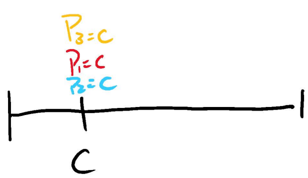
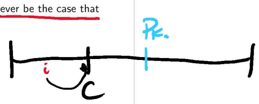
$$BR_1(p_2,p_3)=\begin{cases} p^m & \text{if } \min\{p_2,p_3\}>p^m,\\ \min\{p_2,p_3\}-\varepsilon & \text{if } c<\min\{p_2,p_3\}\le p^m,\\ [c,+\infty) & \text{if } c=\min\{p_2,p_3\},\\ (\min\{p_2,p_3\},+\infty) & \text{if } c>\min\{p_2,p_3\}.\end{cases}$$

- (c,c,c) is indeed a pure strategy Nash equilibrium as in the two firm case



Bertrand Competition - 3 firms

- If (p_1,p_2,p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1,p_2,p_3\}<c$



Bertrand Competition - 3 firms

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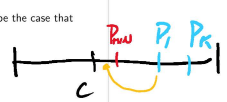

Bertrand Competition - 3 firms

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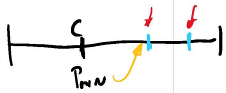
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$



Bertrand Competition - 3 firms

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- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c?

Bertrand Competition - 3 firms

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Bertrand Competition - 3 firms

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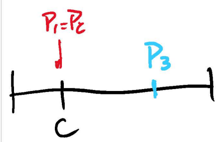
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost



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- Set of all pure strategy Nash equilibria are given by:

$$\boxed{\{(c, c, c+\varepsilon) : \varepsilon \geq 0\} \cup \{(c, c+\varepsilon, c) : \varepsilon \geq 0\} \cup \{(c+\varepsilon, c, c) : \varepsilon \geq 0\}.}$$

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued
Cournot - Revisited
Bertrand Competition
Bertrand Competition - Different costs
Bertrand Competition - 3 Firms
Hotelling and Voting Models

Hotelling

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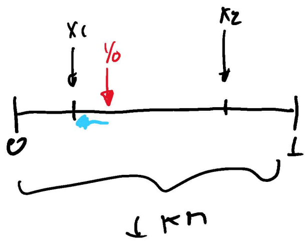
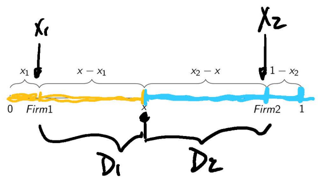
- The game consists of the two players $i = 1, 2$, each of whom chooses a point $x_1, x_2 \in [0, 1]$ simultaneously.



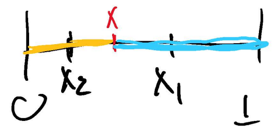
Hotelling



$$x - x_1 = x_2 - x$$

$$2x = x_1 + x_2$$

$$\boxed{x = \frac{x_1 + x_2}{2}}$$



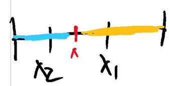
Hotelling

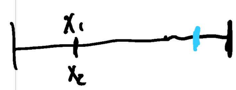
Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Similarly,

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$





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Compute the best response functions

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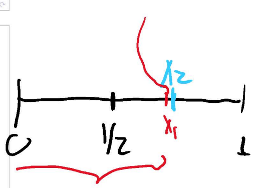
This utility function has a discontinuity at $x_1 = x_2$ and jumps down to $1/2$ at $x_1 = x_2$. There will be no best response for firm 1 (try to set as close to the left the other firm as possible)



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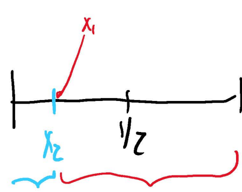
- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)



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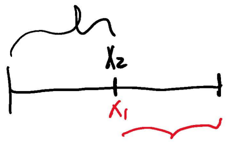
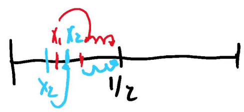
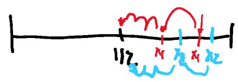
- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)
- **Case 3:** Suppose next that $x_2 = 1/2$. Here there will be a best response for firm 1 at $1/2$







Hotelling

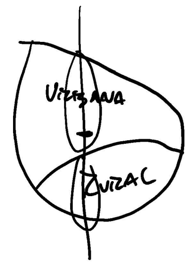
VACIO

$$BR_1(x_2) = \begin{cases} \emptyset & \text{if } x_2 > 1/2 \\ 1/2 & \text{if } x_2 = 1/2 \\ \emptyset & \text{if } x_2 < 1/2. \end{cases}$$

Symmetrically, we have:

$$BR_2(x_1) = \begin{cases} \emptyset & \text{if } x_1 > 1/2 \\ 1/2 & \text{if } x_1 = 1/2 \\ \emptyset & \text{if } x_1 < 1/2. \end{cases}$$

The unique Nash equilibrium is for each firm to choose $\boxed{(x_1, x_2) = (1/2, 1/2)}$. Each firm essentially locates in the same place



Hotelling

- Hotelling can also be done in a discreet setting
- Hotelling can be applied to a variety of situations (e.g., voting)
- But this predicts the opposite of polarization
- With three candidates, predictions are quite different
- All candidates picking $\frac{1}{2}$ is no longer a Nash equilibrium
- What are the set of pure strategy equilibria here? (this is a difficult problem).