Lecture 14: Game Theory // Nash equilibrium

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Mixed strategies

Examples

Mixed strategies

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Consider rock/paper/scissors

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

- This game is entirely stochastic (ability has nothing to do with your chances of winning)

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- Thus, people *tend* choose randomly which of the three options to play
- We would like the concept of Nash equilibrium to reflect this

Mixed strategies

Definition

A mixed strategy σ_i is a function $\sigma_i : S_i \to [0,1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

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- We will denote the set of all mixed strategies of player i by Σ_i

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$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$

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- For instance, assume my opponent is playing randomizing over paper and scissors with probability $\frac{1}{2}$ (i.e., $\sigma_{-i} = (0, \frac{1}{2}, \frac{1}{2})$)
- The expected utility of playing "rock" is
$$E(U_i(rock, \sigma_{-i})) = -1\frac{1}{2} + 1\frac{1}{2} = 0$$
- If I'm randomizing over rock and scissors (i.e., $\sigma_i = (\frac{1}{2}, 0, \frac{1}{2})$) then
$$E(U_i(\sigma, \sigma_{-i})) = \underbrace{-1\frac{1}{4}}_{\text{rock vs paper}} + \underbrace{1\frac{1}{4}}_{\text{rock vs scissors}} + \underbrace{1\frac{1}{4}}_{\text{scissors vs paper}} + \underbrace{0\frac{1}{4}}_{\text{scissors vs scissors}} = \frac{1}{4}$$

Mixed strategies

Definition

A (possibly mixed) strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n)^*$ is a Nash equilibrium if and only if for every i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Sigma_i$.

Mixed strategies

Definition (Mixed Strategy Dominance Definition A)

Let σ_i, σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if for all mixed strategies of the opponents, σ_{-i},

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}).$$

Mixed strategies

If σ_i is better than σ_i' no matter what **pure strategy** opponents play, then σ_i is also strictly better than σ_i' no matter what **mixed strategies** opponents play

Theorem

Let σ_i and σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if and only if for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

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- Then for all $s_{-i} \in S_{-i}$,

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Proof - Part 2

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- For any σ_{-i},

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) \end{aligned}$$

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- So

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) > \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i', s_{-i}) = u_i(\sigma_i', \sigma_{-i})$$

Mixed strategies

Definition (Mixed Strategy Dominance Definition B)

Let σ_i, σ_i' be two mixed strategies of player i. Then σ_i strictly dominates σ_i' if for all pure strategies of the opponents, $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}).$$

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Battle of the sexes

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G	2,1	0,0
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- There are two pure strategy equilibria (G, G) and (P, P)
- We now look for Nash equilibria that involve randomizationby the players

Battle of the sexes

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- **Case 3:** If $q < 1/3$, then $2q < 2/3 < 1 - q$ and therefore the best response is $\lambda = 0$
- Thus, the best response function is given by:

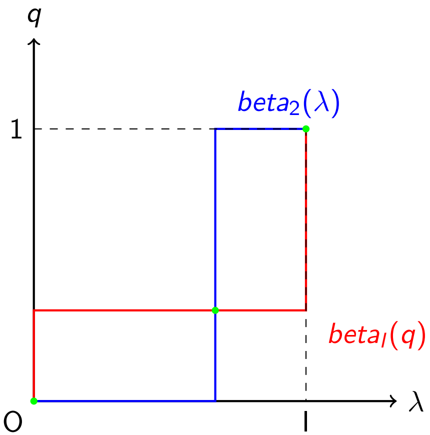
$$BR_1(q) = \begin{cases} 1 & \text{if } q > 1/3 \\ [0,1] & \text{if } q = 1/3 \\ 0 & \text{if } q < 1/3. \end{cases}$$

Battle of the sexes

Similarly we can calculate the best response function for player 2 and we get:

$$BR_2(\lambda) = \begin{cases} 1 & \text{if } \lambda > 2/3 \\ [0,1] & \text{if } \lambda = 2/3 \\ 0 & \text{if } \lambda < 2/3. \end{cases}$$

Battle of the sexes



- There are three points where the best response curves cross: $(1,1),(0,0,),(\frac{2}{3},\frac{1}{3})$

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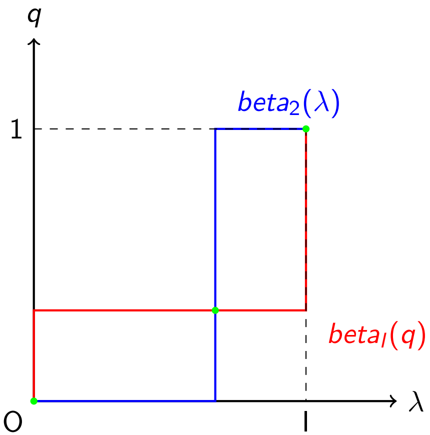

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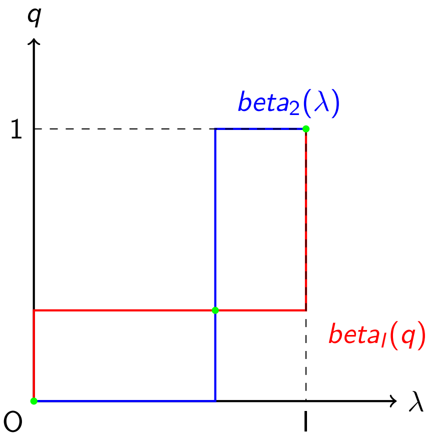

- There are three points where the best response curves cross: $(1, 1), (0, 0,), (\frac{2}{3}, \frac{1}{3})$
- First two are the pure strategy NE we had found before
- Last is a strictly mixed NE: both players randomize

Consider the following game

	E	F	G
A	5, 10	5, 3	3, 4
B	1, 4	7, 2	7, 6
C	4, 2	8, 4	3, 8
D	2, 4	1, 3	8, 4

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- Then $BR_2(\sigma_1) = \{(p, 0, 1-p), p \in [0, 1]\}$

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- D dominates B (player 1)

Reduced game

	E	G
A	5, 10	3, 4
C	4, 2	3, 8
D	2, 4	8, 4

- Note that $\sigma_1 = (p, 0, 1-p)$ with $p > \frac{2}{3}$ dominates C
- $\mathbb{E}U(\sigma_1, E) = 5p + 2(1-p) = 3p + 2$
- $\mathbb{E}U(\sigma_1, G) = 3p + 8(1-p) = 8 - 5p$
- $$\begin{aligned} \mathbb{E}U(\sigma_1, E) &> U(C, E) \\ 3p + 2 &> 4 \\ p &> \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbb{E}U(\sigma_1, G) &> \mathbb{E}U(C, G) \\ 8 - 5p &> 3 \\ p &< \frac{5}{5} = 1 \end{aligned}$$

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- $8 - 6q > 2q + 3$ if $\frac{5}{8} > q$
- $8 - 6q < 2q + 3$ if $\frac{5}{8} < q$
- Thus

$$BR_1(q, 1-q) = \begin{cases} \sigma_1 = (0, 1) & \text{if } 0 \leq q < \frac{5}{8} \\ \sigma_1 = (1, 0) & \text{if } \frac{5}{8} < q \leq 1 \\ \sigma_1 = (p, 1-p) & \text{if } \frac{5}{8} = q \end{cases}$$

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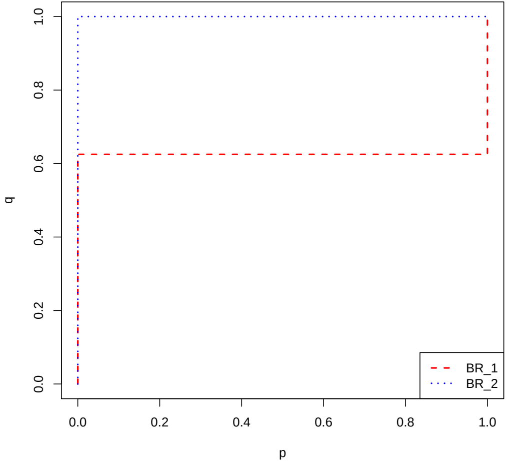
$$BR_2(p, 1-p) = \begin{cases} \sigma_2 = (1, 0) & \text{if } p > 0 \\ \sigma_2 = (q, 1-q) & \text{if } p = 0 \end{cases}$$

Best responses



$NE = \{(A, E), (D, \sigma_2^q)\}$ where $\sigma_2^q = (q, 1 - q)$ and $0 \leq q \leq \frac{5}{8}$