Lecture 16

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Lecture16

Lecture 16: Applications of Subgame Perfect Nash Equilibrium

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Lecture 16: Applications of Subgame Perfect Nash Equilibrium

Ultimatum Game

Alternating offers

Stackelberg Competition

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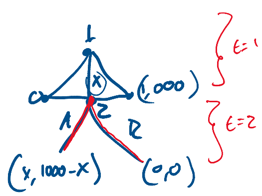
1. Player 1 makes a proposal $(x, 1000-x)$ of how to split 1000 pesos among $(100, 900), \ldots, (800, 200), (900, 100)$
2. Player 2 accepts or rejects the proposal
3. If player 2 rejects both obtain 0. If 2 accepts, then the payoffs or the two players are determined by $(x, 1000-x)$



J_2: A si $x \leq 1000$

J_1: $x = 1{,}000$

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- In any SPNE, player 1 makes the proposal $(900, 100)$

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- When extreme offers like $(900, 100)$ are made, player 2 rejects in many cases
- Player 2 may care about inequality or positive utility associated with "punishment" aversion

$X = 1000$

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Stackelberg Competition

- Two players are deciding how to split a pie of size 1

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- The players would rather get an agreement today than tomorrow (i.e., discount factor)

- Player 1 makes an offer θ_1

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- If player 1 accepts or rejects the proposal
- If player 1 rejects, player 1 makes an offer θ_3
- ... and on and on for T periods

- Player 1 makes an offer θ_1 → $(\theta_1, 1-\theta_1)$
- Player 2 accepts or rejects the proposal
- If player 2 rejects, player 2 makes an offer θ_2
- If player 1 accepts or rejects the proposal
- If player 1 rejects, player 1 makes an offer θ_3
- ... and on and on for T periods
- If no offer is ever accepted, both payoffs equal zero

The discount factor is $\boxed{\delta \leq 1}$.
If Player 1 offer is accepted by Player 2 in round m,

$$\pi_1 = \delta^m \theta_m,$$
$$\pi_2 = \delta^m (1 - \theta_m).$$

If Player 2 offer is accepted, reverse the subscripts

- Consider first the game without discounting $\boxed{\delta = 1}$

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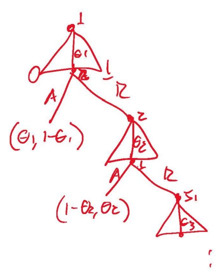

J1 ACABA
⇓
J1 SE QUEDA CON TODO → (1,0)

J2 ACABA
J2 SE QUEDA CON TODO → (0,1)

- Consider first the game without discounting
- There is a unique SPNE: The player that makes the last offer gets the whole pie
- Last-mover advantage

- In the game with discounting, the total value of the pie is 1 in the first period, δ in the second, and so forth

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- In period $(T-1)$, Player 2 could offer Smith δ, keeping $(1-\delta)$ for himself

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- In period $(T-1)$, Player 2 could offer ~~Smith~~ S1 δ, keeping $(1-\delta)$ for himself
- Player 1 would accept (indifferent between accepting and rejecting) since the **whole pie** in the next period is worth δ

$T \to S_1 \to (1,0)\,\delta^T$

$(T-1) \to S_2 \to (x, 1-x)\,\delta^{T-1}$

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- Player 2 would accept (indifferent between accepting and rejecting)
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- Player 1 would accept (indifferent between accepting and rejecting) since the **whole pie** in the next period is worth δ

$$(T-1) \to S_2 \to (x, 1-x)\delta$$

S_1 A si $\cancel{\delta^{T-1}}x \geq \delta^{\cancel{T}}$

$$x \geq \delta$$

$$\hookrightarrow x = \delta$$

$$(\delta, 1-\delta)\delta^{T-1} = (\delta^{T}, (1-\delta)\delta^{T-1})$$

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself

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- Player 2 would accept since he can earn $(1-\delta)$ in the next period, which is worth $\delta(1-\delta)$ today

$$(T-2) \to S_1 \to (1-x, x)\delta^{T-2}$$

S_2 A si $x\cancel{\delta^{T-2}} \geq (1-\delta)\cancel{\delta^{T-1}}$

$$x \geq (1-\delta)\delta$$

$$x = (1-\delta)\delta$$

$$(1-(1-\delta)\delta, (1-\delta)\delta)\delta^{T-2}$$

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself
- Player 2 would accept since he can earn $(1-\delta)$ in the next period, which is worth $\delta(1-\delta)$ today
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$$x \geq (1-(1-\delta)\delta)\delta$$

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- Player 1 would accept...
- ...
- In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting

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$T=1$

Table 1 shows the progression of Player 1's shares when $\delta = 0.9$.

- In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting

$T \geq 1$

Table 1 shows the progression of Player 1's shares when $\delta = 0.9$.

Table 1: Alternating Offers over Finite Time

	Round	1's share	2's share	Total value	Who offers?
0	$T-3$	$\boxed{\delta(1-\delta(1-\delta))}$	$\boxed{1-\delta(1-\delta(1-\delta))}$	δ^{T-4}	2
1	$T-2$	$1-\delta(1-\delta)$	$\delta(1-\delta)$	δ^{T-3}	1
2	$T-1$	δ	$1-\delta$	δ^{T-2}	2
3	T	1	0	δ^{T-1}	1

$\boxed{T=3 \quad T=4}$

$\boxed{\delta_1 \neq \delta_2}$

- If $T = 3$ (i.e, 1 offers, 2 offers, 1 offers)

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- One offers $\delta(1-\delta)$, 2 accepts in period 1

- Player 1 always does a little better when he makes the offer than when Player 2 does

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- If we consider just the class of periods in which Player 1 makes the offer, Player 1's share falls

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$$\left[(1-(1-\delta)\delta)\delta,\ 1-(1-(1-\delta)\delta)\delta\right]\delta^{T-3}$$

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- Recall back to the model of Cournot duopoly, where two firms set quantities

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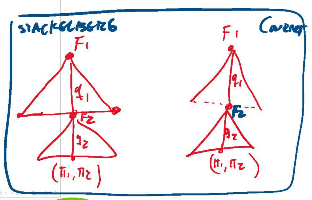
- Recall back to the model of Cournot duopoly, where two firms set quantities
- Suppose instead that the firms move in sequence which is called a **Stackelberg competition game**
- Suppose that the inverse demand function is given by:
$$P(q_1 + q_2).$$
- Firms have the cost functions $c_i(q_i)$.

Te timing of the game is given by:

1. First Firm 1 chooses $q_1 \geq 0$
2. Second Firm 2 observes the chosen q_1 and then chooses q_2

- The game tree in this game is then depicted by an infinite tree



- Let us write down the normal form representation of this game.

$S_2 = \triangleright f : \mathbb{R}_+ \to \mathbb{R}_+$

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- A pure strategy for firm 1 is just a choice of $q_1 \geq 0$
- A strategy for firm 2 specifies what it does after every choice of q_1
- Firm 2's strategy is a function $q_2(q_1)$ which specifies exactly what firm 2 does if q_1 is the chosen strategy of player 1

The utility functions for firm i when firm 1 chooses q_1 and firm 2 chooses the strategy (or function) $q_2(\cdot)$ is given by:

$$\pi_1(q_1, q_2(\cdot)) = P(q_1 + \boxed{q_2(q_1)})q_1 - c_1(q_1)$$
$$\pi_2(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_2(q_1) - c_2(q_2(q_1))$$

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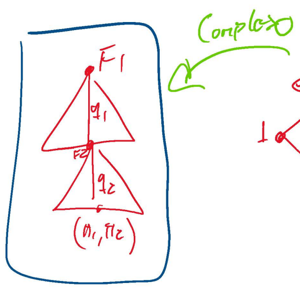
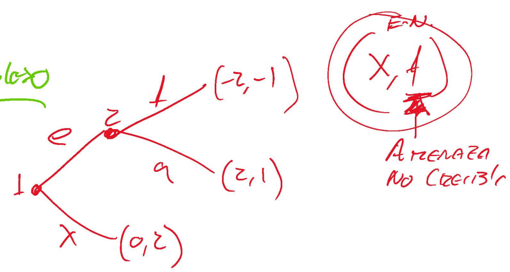
- There are many Nash equilibria of this game which are a bit counterintuitive
- Cconsider the following specific game with demand function given by:

$$P(q_1 + q_2) = A - q_1 - q_2.$$

- Let the marginal costs of both firms be zero
- Then the normal form simplifies:

$$\pi_1: u_1(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_1,$$
$$\pi_2: u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$





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- Let $\alpha \in [0, A)$ and consider the following strategy profile:

$$\boxed{q_1^* = 0}, \boxed{q_2^*(q_1) = \begin{cases} A & \text{if } q_1 \neq 0, \to P = 0 \\ \frac{A}{2} & \text{if } q_1 = 0. \end{cases}}$$

→ Monopolista!

$$P = A - q_1 - q_2$$

$$\boxed{\pi_2^m = (A - q_2)q_2}$$

$$\frac{\partial \pi_2}{\partial q_2} = A - 2q_2 = 0$$

$$\boxed{q_2 = \frac{A}{2}}$$

- What is an example of a Nash equilibrium of this game?
- Let $\alpha \in [0, A)$ and consider the following strategy profile:

$$q_1^* = \alpha, q_2^*(q_1) = \begin{cases} A & \text{if } q_1 \neq \alpha, \\ \frac{A-\alpha}{2} & \text{if } q_1 = \alpha. \end{cases}$$

- Let us check that indeed this constitutes a Nash equilibrium

Firma 1

$$\pi_1(q_1 = 0, q_2(q_1)) = 0$$

$$\pi_1(q_1 > 0, q_2(q_1)) = \underbrace{0}_{\text{Precio}} \cdot q_1 - \cancel{c(q_1)}^{\,=0} = 0$$

Firma 2

$$\boxed{q_1^\circ = 0}$$

$$\pi = (A - q_2)q_2$$

$$\frac{\partial \pi}{\partial q_2} = A - 2q_2 = 0$$

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$$u_1(q_1, q_2^*(\cdot)) = \begin{cases} \left(A - \alpha - \left(\frac{A-\alpha}{2}\right)\right)\alpha > 0 & \text{if } q_1 = \alpha \\ -q_1^2 \leq 0 & \text{if } q_1 \neq \alpha. \end{cases}$$

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- Firm 1 is best responding to player 2's strategy.

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- Thus, firm 2 wants to choose the optimal strategy $q_2(\cdot)$ that maximizes the following utility:

$$\max_{q_2(\cdot)} (A - \alpha - q_2(\alpha))q_2(\alpha)$$

∂q_2

$$\boxed{\frac{A}{2} = q_2}$$

$$\Pi_2 = (A - \underset{\geq 0}{q_1} - q_2)\, q_2$$

$$\frac{\partial \Pi_2}{\partial q_2} = A - q_1 - 2q_2 = 0$$

$$\boxed{\frac{A - q_1}{2} = q_2 \neq A}$$

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$$q_2(\alpha) = \frac{A - \alpha}{2}.$$

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- The utility function of firm 2 does not depend at all on what it chooses for $q_2^*(q_1)$ when $q_1 \neq \alpha$
- In particular, q_2^* is a best response for firm 2

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- In particular, in the Nash equilibrium corresponding to $\alpha = 0$, the equilibrium outcome is for firm 1 to choose a quantity of 0 and firm 2 setting a price of $A/2$

- This would be the same outcome if firm 2 were the monopolist in this market

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- If firm 1 were to hypothetically choose $q_1 > 0$, then firm 2 would obtain negative profits if it indeed follows through with $q_2^*(q_1)$.

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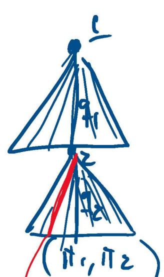
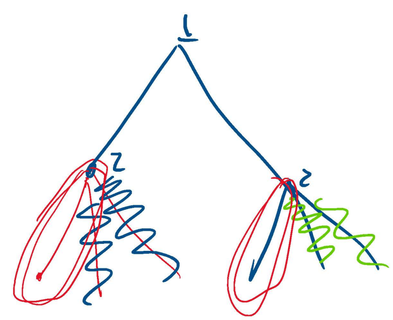
- Lets continue with the setting in which marginal costs are zero and the demand function is given by $A - q_1 - q_2$

- We always start with the smallest/last subgames which correspond to the decisions of firm 2 after firm 1's choice of q_1 has been made



$t=1$

$t=2$



$t=2$

$\boxed{T=2}$

$$\boxed{\pi_2 = (A - q_1 - q_2)\, q_2}$$

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- The utility function of firm 2 is given by:

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- So, player 2 solves:

$$\max_{q_2(\cdot)} (A - q_1 - q_2(q_1))q_2(q_1).$$

- **Case 1:** $q_1 > A$

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- **Case 2:** $q_1 \leq A$
- In this case, the first order condition implies:

$$q_2^*(q_1) = \frac{A - q_1}{2}.$$

$$\boxed{\Pi_2 = (A - q_1 - q_2)q_2}$$

$$\frac{\partial \Pi_2}{\partial q_2} = A - q_1 - 2q_2 = 0$$

$$\boxed{q_2(q_1) = q_2 = \frac{A - q_1}{2}}$$

$$\boxed{T = 1}$$

$$\Pi_1 = (A - q_1 - q_2)q_1$$

INCORPORAMOS $MR_2(q_1)$ EN $t = 2$

$$\Pi_1 = \left(A - q_1 - \left(\frac{A - q_1}{2}\right)\right)q_1$$

$$\Pi_1 = \left(\frac{A - q_1}{2}\right)q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{A - 2q_1}{2} = 0$$

$$\boxed{\frac{A}{2} = q_1^*}$$

EQ. PERFECTO DE

$$EPS = \left(q_1^* = \frac{A}{2}, \; q_2(q_1) = \frac{A - q_1}{2}\right)$$

ESTRATEGIA

- Thus, in any SPNE, player 2 must play the following strategy:

$$q_2^*(q_1) = \begin{cases} \frac{A-q_1}{2} & \text{if } q_1 \leq A \\ 0 & \text{if } q_1 > A. \end{cases}$$

- Then player 1's utility function given that player 2 plays q_2^* is given by:

$$u_1(q_1, q_2^*(\cdot)) = q_1(A - q_1 - q_2^*(q_1)) = \begin{cases} q_1(A - q_1) & \text{if } q_1 > A, \\ q_1 \frac{A-q_1}{2} & \text{if } q_1 \leq A. \end{cases}$$

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- Thus, firm 1 maximizes $\max_{q_1} u_1(q_1, q_2^*(\cdot))$
- Firm 1 will never choose $q_1 > A$ since then it obtains negative profits
- Thus, firm 1 maximizes:

$$\max_{q_1 \in [0,A]} q_1 \frac{A - q_1}{2}.$$

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PERFECTO DE SUB-JUEGOS

ESTRATEGIA DE EQ

$$\boxed{q_2^* \text{ EN EQUILIBRIO} = \frac{A}{4}}$$

NO ES LA ESTRATEGIA

$$q_2(q_1) = \frac{A}{4}$$

$$¿\boxed{EN = \left(q_1 = \frac{A}{2},\ q_2(q_1) = \frac{A}{4}\right)}? \longrightarrow ¡NO!$$

$$\Pi_1 = \left(A - q_1 - \frac{A}{4}\right) q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = A - 2q_1 - \frac{A}{4} = 0$$

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- The **equilibrium outcome** is for firm 1 to choose $A/2$ and firm 2 to choose $A/4$

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- Lets solve for the set of SPNE (which is the same as NE) in the Cournot game with the same demand function and same costs
- In this case, (q_1^*, q_2^*) is a NE if and only if

$$q_1^* \in BR_1(q_2^*), q_2^* \in BR_2(q_1^*).$$

$2q_1$

$$\frac{3A}{4} = 2q_1$$

$$\boxed{\frac{3A}{8} = q_1} \neq \frac{A}{2}$$

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- As a result, solving these two equations, we get:

$$\boxed{q_1^* = q_2^* = \frac{A}{3}.}$$

In the Cournot game, note that firms' payoffs are:

$$\boxed{\pi_1^c = \frac{A^2}{9}, \pi_2^c = \frac{A^2}{9}.}$$

As we already saw, this was not Pareto efficient since each firm is getting a payoff that is strictly less than 1/2 of the monopoly profits.

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$= q_1 = \frac{A}{2}, q_2 = \frac{A}{4}$

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- But by choosing something optimal, firm 1 will be able to do even better