Lecture 17: Applications of Subgame Perfect Nash Equilibrium

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- We will use (G, T) to denote that game G is repeated T times

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3. This game proceeds until time T.
4. After time T, if the action profiles chosen in times $1, 2, \ldots, T$ are given by $\left((a_i^1, a_{-i}^1), \ldots, (a_i^T, a_{-i}^T)\right)$:
$$\sum_{t=1}^{T} \delta^{t-1} u_i(a_i^t, a_{-i}^t).$$

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- Working incurs a cost of 1 however increases the utility of the other player $-i$ by 2
- Thus,

$$u_i(e_i, e_{-i}) = 2e_{-i} - e_i.$$

Prisoner's Dilemma (Game G)

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

- What happens when $T = 1$

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- NE: Players 1 and 2 will both choose $(e_1 = 0, e_1 = 0)$

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1. Both players play the simultaneous move game G.
2. Both players observe the actions chosen by the two players. Then they play G again.
3. Then payoffs are realized as the discounted sum of the utilities of the actions in each period with discount factor $\delta \in (0, 1]$.

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$$u_1 = 1 + \delta \cdot 2$$
$$u_2 = 1 + \delta \cdot (-1).$$

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- Player 1 has 5 information sets in total
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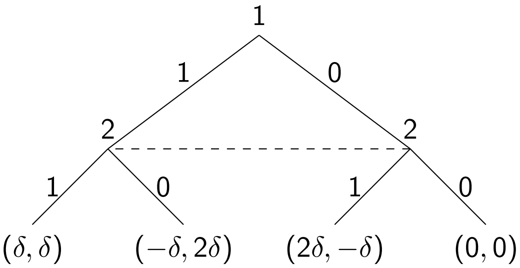
- Player 1 has a total of 32 (2^5) pure strategies
- Similarly, player 2 has a total of 32 pure strategies

- There are 5 subgames



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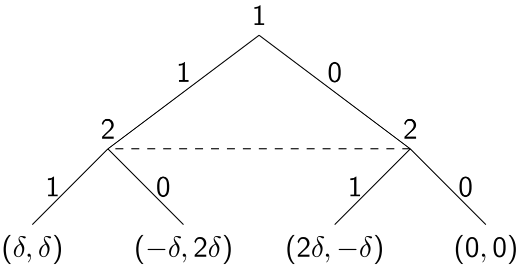
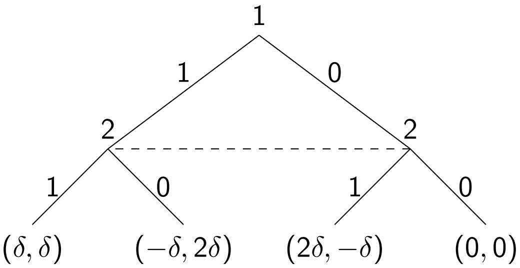


- There are 5 subgames
- Start at the end of the game (i.e., $T = 2$)
- The first subgame that we will analyze is the one that the players encounter after having play $(e_1^1 = 0, e_2^1 = 0)$ in $T = 1$:



The Nash equilibria can be seen by writing out the normal form of the game.

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	δ, δ	$-\delta, 2\delta$
$e_1 = 0$	$2\delta, -\delta$	$0, 0$

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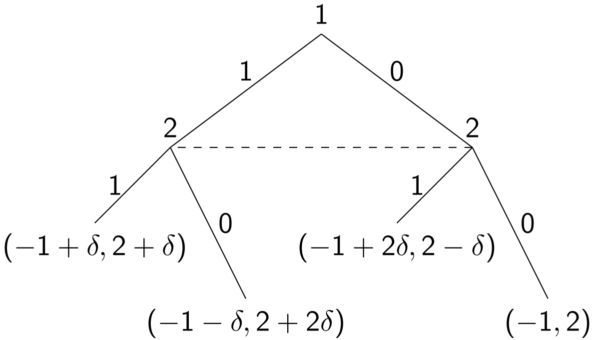
- Therefore after having observed $(e_1^1 = 0, e_2^1 = 0)$ in the first period, both players will play $(e_1^2 = 0, e_2^2 = 0)$ in period 2

Consider the subgame following a play of $(e_1^1 = 1, e_2^1 = 0)$ in the first period. The extensive form of this subgame is given by:



The normal form of this subgame can be seen in the Table

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1 + \delta, 2 + \delta$	$-1 - \delta, 2 + 2\delta$
$e_1 = 0$	$-1 + 2\delta, 2 - \delta$	$-1, 2$

- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium

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- In any SPNE, $(e_1^2 = 0, e_2^2 = 0)$ must be played after observing $(e_1^1 = 1, e_2^1 = 0)$

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames
- Regardless of the observed action, $(0, 0)$ is played in period 2
- Why is this the case?
- The idea is that payoffs that have accrued in period 1 are essentially sunk, and have no influence on incentives in period 2

To see this consider the normal form representation in the subgame after the observation of $(e_1^1 = 1, e_2^1 = 0)$

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1 + \delta, 2 + \delta$	$-1 - 1\delta, 2 + 2\delta$
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- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by δ to obtain the following payoff matrix

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- We can do the same thing for player 2's payoffs and get the payoff matrix

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	$e_2 = 1$	$e_2 = 0$
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- This will be true no matter the action profile played in period 1

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- Both players play $(e_1^2 = 0, e_2^2 = 0)$ after any information set in the last period

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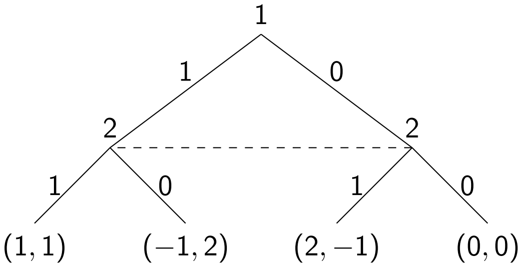
- Both players anticipate that $(e_1^2 = 0, e_2^2 = 0)$ will be played after any chosen action profile in the first period
- We can simplify the extensive form game to the following:



If we draw the normal form of this game, then we get:

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

The unique Nash equilibrium of the above normal form game is $(e_1^1 = 0, e_2^1 = 0)$

Therefore the unique SPNE is:

$$\left(\begin{pmatrix} e_1^1 = 0 & \begin{array}{l} e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \end{array} \end{pmatrix}, \begin{pmatrix} e_2^1 = 0 & \begin{array}{l} e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \end{array} \end{pmatrix} \right)$$

In other words both players always shirk

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- Whenever the stage game has a unique NE, then the only SPNE of a **finite horizon** repeated game with that stage game is the repetition of the stage game NE

Theorem

Suppose that the stage game G has exactly one NE, $(a_1^*, a_2^*, \ldots, a_n^*)$. Then for any $\delta \in (0, 1]$ and any T, the T-times repeated game has a unique SPNE in which all players i play a_i^* at all information sets.

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- Knowing that the stage game Nash equilibrium is going to be played tomorrow, at any information set, we can ignore the past payoffs
- We concentrate just on the payoffs in the future. Thus in period $T-1$, player i simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \ldots, a_n^{T-1} = a_n^*$.
- Following exactly this induction, we can conclude that every player must play a_i^* at all times and all histories