# Lecture 17

Thursday, April 22, 2021 6:00 PM

Lecture17

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## Lecture 17: Applications of Subgame Perfect Nash Equilibrium

Mauricio Romero

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### Lecture 17: Repeated Games

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- We will use $\boxed{(G, T)}$ to denote that game $G$ is repeated $\boxed{T}$ times

JUEGO BASE

1. In period 1, players simultaneously play the game $G$.

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2. Players observe the actions chosen by the players in period 1. Then in period 2, players simultaneously play the game $G$.
3. This game proceeds until time $T$.
4. After time $T$, if the action profiles chosen in times $1, 2, \ldots, T$ are given by $((a_i^1, a_{-i}^1), \ldots, (a_i^T, a_{-i}^T))$:

$$\boxed{\sum_{t=1}^{T} \delta^{t-1} u_i(a_i^t, a_{-i}^t).} = \text{Valor Presente } \Sigma \text{ Pagos}.$$

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- Each player $i = 1, 2$ simultaneously decide whether to play $e_i = 1$ (work) or $e_i = 0$ (shirk)

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- Working incurs a cost of 1 however increases the utility of the other player $-i$ by 2
- Thus,

$$\boxed{u_i(e_i, e_{-i}) = 2e_{-i} - e_i.}$$

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Prisoner's Dilemma (Game $G$)

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | 1, 1 | $-1, 2$ |
| $e_1 = 0$ | $2, -1$ | 0, 0 |

EN: $(e_1 = 0, e_2 = 0)$

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- What happens when $T = 1$

- What happens when $T = 1$
- NE: Players 1 and 2 will both choose $(e_1 = 0, e_1 = 0)$

Imagine players are engaged in a long run relationship that lasts more than just playing the game once: $(G, 2)$

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1. Both players play the simultaneous move game $G$.
2. Both players observe the actions chosen by the two players. Then they play $G$ again.
3. Then payoffs are realized as the discounted sum of the utilities of the actions in each period with discount factor $\delta \in (0, 1]$.

Suppose that the two players chose $(e_1 = 1, e_2 = 1)$ in the first period
In the second period, they chose $(e_1 = 0, e_2 = 1)$

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$$u_1 = 1 + \delta \cdot 2$$
$$u_2 = 1 + \delta \cdot (-1).$$

- We will solve for the set of pure SPNE of this game.

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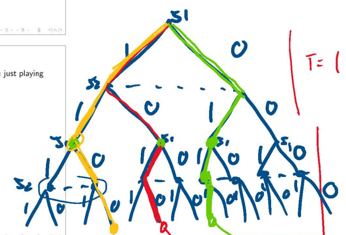
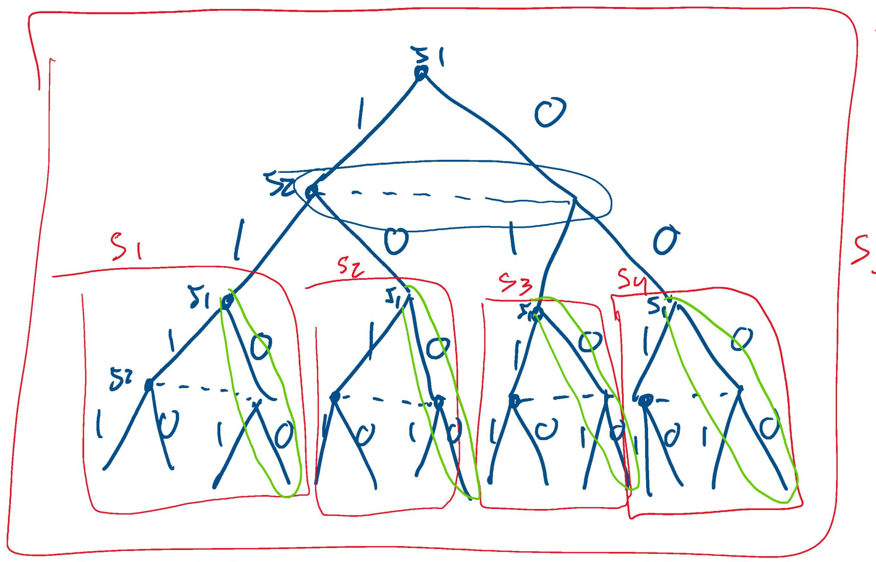



$U_1 = 1 + 0\delta$
$U_2 = 1 + 0\delta$

$U_1 = -1 + (-1)\delta$
$U_2 = 2 + (2)\delta$

$U_1 = 2 + 1\delta$
$U_2 = -1 + 1\delta$



5 SUBJUEGOS

$S_5$

$\boxed{S_1 - S_4}$

| $S_1$ \ $S_2$ | 1 | 0 |
|---|---|---|
| 1 | $U^1_{t=1} + 1\delta,\ U^2_{t=1} + 1\delta$ | $U^1_{t=1} - 1\delta,\ U^2_{t=1} + 2\delta$ |
| 0 | $U^1_{t=1} + 2\delta,\ U^2_{t=1} - 1\delta$ | $U^1_{t=1} + 0\delta,\ U^2_{t=1} + 0\delta$ |

$(U^1_{t=1}, U^2_{t=1}) +$

| | 1 | 0 |
|---|---|---|
| 1 | $1\delta, 1\delta$ | $-1\delta, 2\delta$ |
| 0 | $2\delta, -1\delta$ | $0\delta, 0\delta$ |

$(U^1_{t=1}, U^2_{t=1}) + \delta \Big($

| | 1 | 0 |
|---|---|---|
| 1 | 1, 1 | -1, 2 |
| 0 | 2, -1 | 0, 0 |

$\Big)$

$2 \times 2 \times 2 \times 2 \times 2 = \boxed{2^5}$

$S'_1 = \{(\ ,\ ,\ ,\ ,\ )\}$

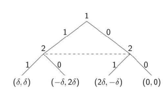
t=1 $S_1$ $S_2$ $S_3$ $S_4$

$S'_2 =$ MUY PARECIDO

- We will solve for the set of pure SPNE of this game.
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 ($2^5$) pure strategies

- We will solve for the set of pure SPNE of this game.
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 ($2^5$) pure strategies
- Similarly, player 2 has a total of 32 pure strategies

- There are 5 subgames



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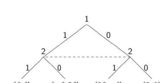
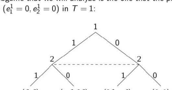
- Start at the end of the game (i.e., $T = 2$)



- There are 5 subgames
- Start at the end of the game (i.e., $T = 2$)
- The first subgame that we will analyze is the one that the players encounter after having play ($e_1^1 = 0, e_2^1 = 0$) in $T = 1$:



The Nash equilibria can be seen by writing out the normal form of the game.

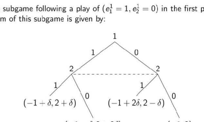
Normal Form of Extensive Form

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | $\delta, \delta$ | $-\delta, 2\delta$ |
| $e_1 = 0$ | $2\delta, -\delta$ | $0, 0$ |

- This game has a unique Nash equilibrium in which the players play ($e_1^2 = 0, e_2^2 = 0$)

- This game has a unique Nash equilibrium in which the players play ($e_1^2 = 0, e_2^2 = 0$)
- Therefore after having observed ($e_1^1 = 0, e_2^1 = 0$) in the first period, both players will play ($e_1^2 = 0, e_2^2 = 0$) in period 2

Consider the subgame following a play of ($e_1^1 = 1, e_2^1 = 0$) in the first period. The extensive form of this subgame is given by:



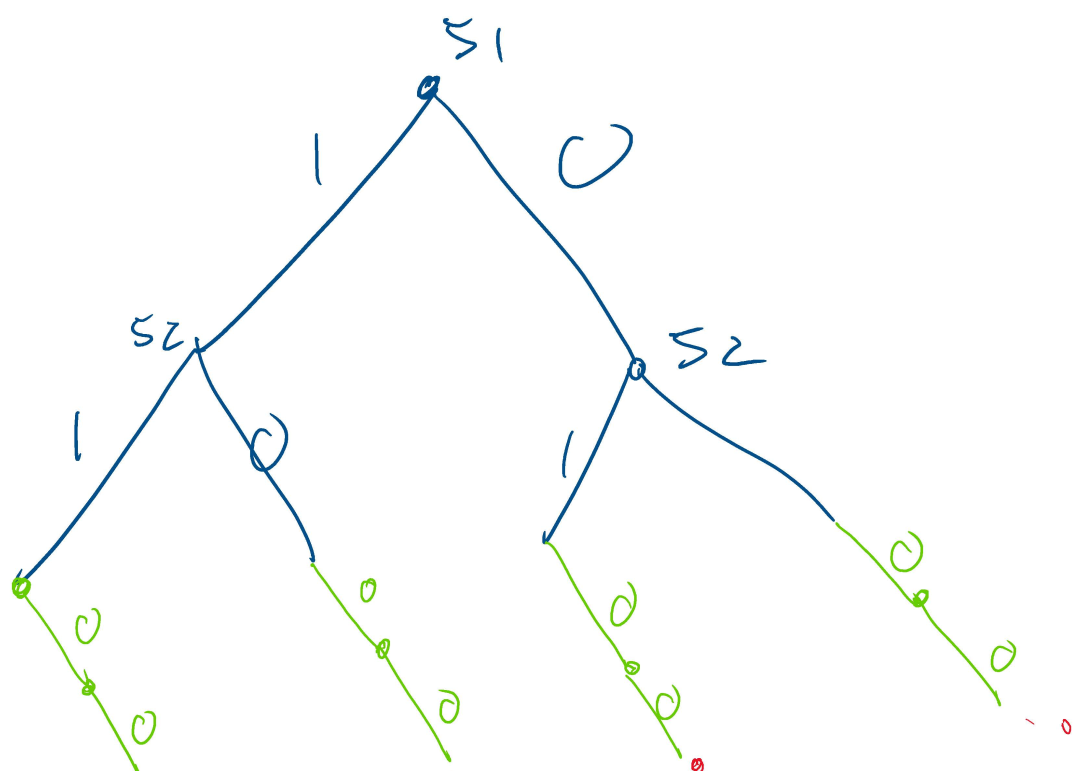
$$(U_{t=1}, U_{t=1}) + \delta \begin{array}{c|c|c} 1 & 1,1 & -1,2 \\ \hline 0 & 2,-1 & 0,0 \end{array}$$

$$(U^1_{t=1}, U^2_{t=1}) + \delta \begin{array}{|c|c|} 1,1 & -1,2 \\ \hline 2,-1 & 0,0 \end{array} \sim$$

MISMOS INCENTIVOS DE G

$EN_{SUBJUEGO} = (e_1 = 0, e_2 = 0)$

$\boxed{T=1} \rightarrow \boxed{S_5}$ JUEGO COMPLETO



$(1+0\delta, 1+0\delta)$ $(-1+0\delta, 2+0\delta)$ $(2+0\delta, -1+0\delta)$ $(0+0\delta, 0+0\delta)$

T=1

PAGO EN T=2

$(-1-\delta, 2+2\delta)$ $(-1, 2)$

The normal form of this subgame can be seen in the Table

Normal Form of Extensive Form

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | $-1+\delta, 2+\delta$ | $-1-\delta, 2+2\delta$ |
| $e_1 = 0$ | $-1+2\delta, 2-\delta$ | $-1, 2$ |

- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium

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- In any SPNE, $(e_1^2 = 0, e_2^2 = 0)$ must be played after observing $(e_1^1 = 1, e_2^1 = 0)$

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames
- Regardless of the observed action, $(0, 0)$ is played in period 2
- Why is this the case?

- We can go through the remaining smaller subgames after the observation of $(e_1^1 = 1, e_2^1 = 0)$ and after the observation of $(e_1^1 = 1, e_2^1 = 1)$
- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames
- Regardless of the observed action, $(0, 0)$ is played in period 2
- Why is this the case?
- The idea is that payoffs that have accrued in period 1 are essentially sunk, and have no influence on incentives in period 2

To see this consider the normal form representation in the subgame after the observation of $(e_1^1 = 1, e_2^1 = 0)$

Normal Form of Extensive Form

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | $-1+\delta, 2+\delta$ | $-1-1\delta, 2+2\delta$ |
| $e_1 = 0$ | $-1+2\delta, 2-\delta$ | $-1, 2$ |

$S_1$

| | 1 | 0 |
|---|---|---|
| 1 | $1+0\delta, 1+0\delta$ | $-1+0\delta, 2+0\delta$ |
| 0 | $2+0\delta, -1+0\delta$ | $0+0\delta, 0+0\delta$ |

$\rightarrow$

$T=1$

| | | |
|---|---|---|
| | $1, 1$ | $-1, 2$ |
| | $2, -1$ | $0, 0$ |

$+ \; \underbrace{(0,0)\delta}$ $\sim$ Mismos incentivos juego base

$EN_{SS}: (e_1 = 0, e_2 = 0)$

$\underline{EPS}: \left( S_1 : (0,0,0,0,0) \quad S_2 : (0,0,0,0,0) \right)$

- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by $\delta$ to obtain the following payoff matrix

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- We can do the same thing for player 2's payoffs and get the payoff matrix

Normal Form of Extensive Form

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | $1, 1$ | $-1, 2$ |
| $e_1 = 0$ | $2, -1$ | $0, 0$ |

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- This will be true no matter the action profile played in period 1

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- Basically after any history, the strategic normal form is essentially the same as the original prisoner's dilemma
- Both players play $(e_1^2 = 0, e_2^2 = 0)$ after any information set in the last period

- Now let us see what must be played in the first period by the two players

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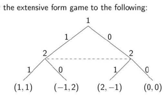
- Both players anticipate that $(e_1^2 = 0, e_2^2 = 0)$ will be played after any chosen action profile in the first period

- Now let us see what must be played in the first period by the two players
- Both players anticipate that $(e_1^2 = 0, e_2^2 = 0)$ will be played after any chosen action profile in the first period
- We can simplify the extensive form game to the following:




If we draw the normal form of this game, then we get:

Normal Form of Extensive Form

| | $e_2 = 1$ | $e_2 = 0$ |
|---|---|---|
| $e_1 = 1$ | $1, 1$ | $-1, 2$ |
| $e_1 = 0$ | $2, -1$ | $0, 0$ |

The unique Nash equilibrium of the above normal form game is $(e_1^1 = 0, e_2^1 = 0)$

Therefore the unique SPNE is:

$$\left(\left(e_1^1 = 0 \quad \begin{array}{c} e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \end{array}\right), \left(e_2^1 = 0 \quad \begin{array}{c} e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \end{array}\right)\right)$$

In other words both players always shirk

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EPS

- Here the unique SPNE requires all players to play $e_i = 0$ at all periods and all information sets
- Thus, the equilibrium outcome is simply the repetition of the unique NE of the stage game
- This holds more generally when the stage game has a $\boxed{\text{unique NE}}$ (G,T) ↳ FINITO
- Whenever the stage game has a unique NE, then the only SPNE of a **finite horizon** repeated game with that stage game is the repetition of the stage game NE

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Theorem
*Suppose that the stage game $G$ has exactly one NE, $\boxed{(a_1^*, a_2^*, \ldots, a_n^*)}$. Then for any $\delta \in (0,1]$ and any $T$, the $T$-times repeated game has a unique SPNE in which all players $i$ play $a_i^*$ at all information sets.*

"EN TODOS LOS PERIODOS"

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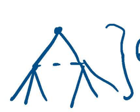
- In the last period, the incentives of all players are exactly the same as if the game were being played once
- Thus all players must play the stage game Nash equilibrium action regardless of the history of play up to that point
- But then we can induct
- Knowing that the stage game Nash equilibrium is going to be played tomorrow, at any information set, we can ignore the past payoffs
- We concentrate just on the payoffs in the future. Thus in period $T-1$, player $i$ simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$



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- What player $i$ plays today has no consequences for what happens in period $T$ since we saw that all players will play $a^*$ no matter what happens in period $T-1$

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \dots, a_n^{T-1} = a_n^*$.

- Following exactly this induction, we can conclude that every player must play $a_i^*$ at all times and all histories