Lecture 18

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Lecture18

Lecture 18: Repeated Games

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Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

Theorem

Suppose that the stage game G has exactly one NE, $(a_1^*, a_2^*, \ldots, a_n^*)$. Then for any $\delta \in (0, 1]$ and any T, the T-times repeated game has a unique SPNE in which all players i play a_i^* at all information sets.

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- But then we can induct
- Knowing that the stage game Nash equilibrium is going to be played tomorrow, at any information set, we can ignore the past payoffs
- We concentrate just on the payoffs in the future. Thus in period $T-1$, player i simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \ldots, a_n^{T-1} = a_n^*$.

- Following exactly this induction, we can conclude that every player must play a_i^* at all times and all histories

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Recap from last class

More than one NE in the stage game

Example 1

Example 2

- What would happen if there are more than one NE of the stage game?

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- Suppose instead that the stage game looks as follows

CASTIGO

Normal Form (G)

	A_2	B_2	C_2
A_1	$\boxed{1,1}$	0, 0	0, 0
B_1	0, 0	$\boxed{4,4}$	1, 5
C_1	0, 0	5, 1	$\boxed{3,3}$

PREMIO

EN = {(A₁, A₂); (C₁, C₂)}.

(B_1, B_2) PARETO DOMINA EN

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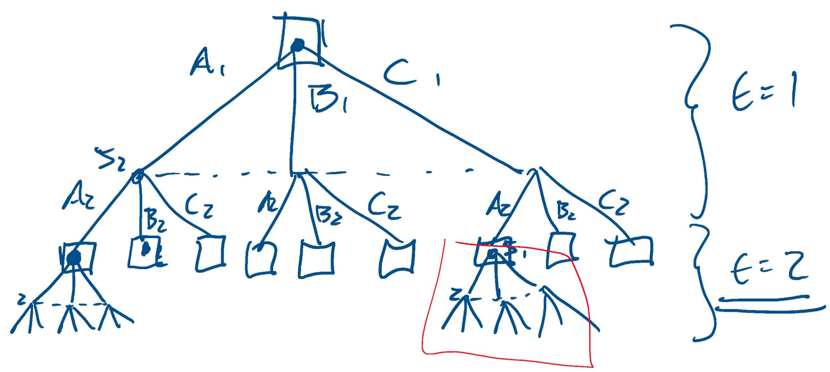
- (B_1, B_2) is not a Nash equilibrium if the game is only played once



- If the game is only played once
- There are two pure strategy Nash equilibria: (A_1, A_2) and (C_1, C_2).
- (B_1, B_2) is not a Nash equilibrium if the game is only played once
- In the one-shot game, the Nash equilibria are inefficient because they are Pareto dominated by (B_1, B_2)

$\boxed{T=2}$ $3 \times 3 \times 3 \cdots \times 3 = 3^{10} = 59{,}049$

$S_1 = (\; , \underbrace{\; , \; , \; , \; , \; , \; , \; , \; , \;}_{9 \text{ CONTINGENCIAS}})$

$t=1$ $\quad$ $t=2$

- Playing the *NE* of the stage game in every period is a SPNE in the repeated game

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- The logic is the same as when there is a single *NE*

- Always playing (A_1, A_2) is a SPNE

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- Player 1's strategy is given by:
  1. Play A_1 in period 1;
  2. Play A_1 at all histories in period 2.
- Player 2's strategy is given by:
  1. Play A_2 in period 1;
  2. Play A_2 at all histories in period 2.

- Always playing (C_1, C_2) is a SPNE

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- Player 1's strategy is given by:
  1. Play C_1 in period 1;
  2. Play C_1 at all histories in period 2.

- Player 2's strategy is given by:
  1. Play C_2 in period 1;
  2. Play C_2 at all histories in period 2.

But are there more?

- Combining *NE* of the stage game is also a SPNE

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- The logic is the same as before

- Playing (A_1, A_2) in $t = 1$ and (C_1, C_2) in $t = 2$ is a SPNE

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- Player 1's strategy is given by:
  1. Play A_1 in period 1;
  2. Play C_1 at all histories in period 2.

- Player 2's strategy is given by:
  1. Play A_2 in period 1;
  2. Play C_1 at all histories in period 2.

- Similarly, playing (C_1, C_2) in $t = 1$ and (A_1, A_2) in $t = 2$ is a SPNE

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- Player 1's strategy is given by:
  1. Play C_1 in period 1;
  2. Play A_1 at all histories in period 2.

- Player 2's strategy is given by:
  1. Play C_2 in period 1;
  2. Play A_1 at all histories in period 2.

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- What makes a repeated game interesting is when players play strategies in SPNE that condition on what happened in the past
- This could not happen when the stage game had a unique NE
- In the last period, all players were required to play the unique NE action after all histories!

IMPORTANTE!!

EPS = (A_1, A_2) EN TODOS LOS PERIODOS ✓

(C_1, C_2) EN " " " ✓

(A_1, A_2) EN $t=1$, (C_1, C_2) EN $t=2$ ✓

(C_1, C_2) EN $t=1$, (A_1, A_2) EN $t=2$ ✓

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- What makes a repeated game interesting is when players play strategies in SPNE that condition on what happened in the past
- This could not happen when the stage game had a unique NE
- In the last period, all players were required to play the unique NE action after all histories! Why?

Proof

- To see this, suppose that a history (a_1, a_2) was played in period 1 resulting in payoffs from period 1 of (x, y)

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- To see this, suppose that a history (a_1, a_2) was played in period 1 resulting in payoffs from period 1 of (x, y)
- Then the normal form of the subgame starting in period 2 is given by:

Normal Form

	A_2	B_2	C_2
A_1	$(x,y)+\delta(1,1)$	$(x,y)+\delta(0,0)$	$(x,y)+\delta(0,0)$
B_1	$(x,y)+\delta(0,0)$	$(x,y)+\delta(4,4)$	$(x,y)+\delta(1,5)$
C_1	$(x,y)+\delta(0,0)$	$(x,y)+\delta(5,1)$	$(x,y)+\delta(3,3)$

Proof

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- The set of Nash equilibria of this subgame is given by (A_1, A_2) and (C_1, C_2)
- Thus after any history, the set of pure strategy NE are (A_1, A_2) or (C_1, C_2)

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- The set of Nash equilibria of this subgame is given by (A_1, A_2) and (C_1, C_2)
- Thus after any history, the set of pure strategy NE are (A_1, A_2) or (C_1, C_2)
- Since SPNE requires Nash equilibrium in every subgame, this means that after any history, (A_1, A_2) or (C_1, C_2) must be played

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- Lets try to find a SPNE in which (B_1, B_2) is played in the first period.

Normal Form

	A_2	B_2	C_2
A_1	$1,1$	$0,0$	$0,0$
B_1	$0,0$	$4,4$	$1,5$
C_1	$0,0$	$5,1$	$3,3$

CASTIGO!

PREMIO!

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  2. Play A_1 in period 2 if anything other than (B_1, B_2) is played in period 1,

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- Anna plays the following strategy:
  1. Play B_1 in period 1.
  2. Play A_1 in period 2 if anything other than (B_1, B_2) is played in period 1,
  3. Play C_1 in period 2 if (B_1, B_2) is played in period 1.

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- Anna plays the following strategy:
  1. Play B_1 in period 1.
  2. Play A_1 in period 2 if anything other than (B_1, B_2) is played in period 1,
  3. Play C_1 in period 2 if (B_1, B_2) is played in period 1.
- Bob plays a similar strategy:

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- Anna plays the following strategy:
  1. Play B_1 in period 1.
  2. Play A_1 in period 2 if anything other than (B_1, B_2) is played in period 1,
  3. Play C_1 in period 2 if (B_1, B_2) is played in period 1.
- Bob plays a similar strategy:
  1. Play B_2 in period 1.

- Consider the following strategy profile, where we punish in $t = 2$ if we don't play (B_1, B_2) in $t = 1$
- Anna plays the following strategy:
  1. Play B_1 in period 1.
  2. Play A_1 in period 2 if anything other than (B_1, B_2) is played in period 1,
  3. Play C_1 in period 2 if (B_1, B_2) is played in period 1.
- Bob plays a similar strategy:
  1. Play B_2 in period 1.
  2. Play A_2 in period 2 if anything other than (B_1, B_2) is played in period 1,

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  3. Play C_1 in period 2 if (B_1, B_2) is played in period 1.
- Bob plays a similar strategy:
  1. Play B_2 in period 1.
  2. Play A_2 in period 2 if anything other than (B_1, B_2) is played in period 1,
  3. Play C_2 in period 2 if (B_1, B_2) is played in period 1.

¿EPS? EN $t=2$ ✓

$\boxed{T=1}$

$U_1(\boxed{A_1}, B_2) + \delta U_1(A_1, A_2) = 0 + \delta$

$\boxed{S1}$ $U_1(B_1, B_2) + \delta U_1(C_1, C_2) \geq U_1(\boxed{C_1}, B_2) + \delta U_1(A_1, A_2) = 5 + \delta$

$4 + 3\delta \geq 5 + \delta$

$2\delta \geq 1$

$\boxed{\delta \geq 1/2}$

$\boxed{S2}$ $\boxed{\text{SIMETRICO}}$

If (B_1, B_2) is observed in the first period, the subgame corresponding to that observation admits the following normal form:

Normal Form

	A_2	B_2	C_2
A_1	$(4,4)+\delta(1,1)$	$(4,4)+\delta(0,0)$	$(4,4)+\delta(0,0)$
B_1	$(4,4)+\delta(0,0)$	$(4,4)+\delta(4,4)$	$(4,4)+\delta(1,5)$
C_1	$(4,4)+\delta(0,0)$	$(4,4)+\delta(5,1)$	$(4,4)+\delta(3,3)$

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- If we add the same utility to all boxes, then the preferences of players are completely unchanged
- Therefore the set of Nash equilibria are the same in this subgame as in the stage game
- So it is a Nash equilibrium in this subgame for players to play (A_1, A_2), which is consistent with the strategy that we proposed

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- Let us now check that after observing $(\alpha_1, \alpha_2) \neq (B_1, B_2)$, then it is a Nash equilibrium in the subgame for players to play (C_1, C_2)

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- If $(\alpha_1, \alpha_2) \neq (B_1, B_2)$ is observed there are some payoffs (x, y) such that the subgame induces the following normal form

Normal Form

	A_2	B_2	C_2
A_1	$(x, y) + \delta(1, 1)$	$(x, y) + \delta(0, 0)$	$(x, y) + \delta(0, 0)$
B_1	$(x, y) + \delta(0, 0)$	$(x, y) + \delta(4, 4)$	$(x, y) + \delta(1, 5)$
C_1	$(x, y) + \delta(0, 0)$	$(x, y) + \delta(5, 1)$	$(x, y) + \delta(3, 3)$

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- Again in this case, note that we are simply adding the same payoff profile (x, y) to every box and multiplying by δ
- Therefore, the Nash equilibrium is again the set of Nash equilibrium of the original stage game
- In this subgame, it is a Nash equilibrium for players to play (A_1, A_2)

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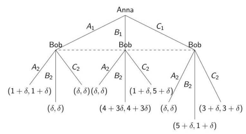
- The only other subgame is the whole game itself
- We need to check that indeed the strategies constitute a Nash equilibrium in the whole game
- To do this, we already specified the play at all information sets in the second period

So we can simplify the game which gives the following game tree.



The normal form of this game (conditional on what happens in $T = 2$) is:

Normal Form

	A_2	B_2	C_2
A_1	$1+\delta, 1+\delta$	δ, δ	δ, δ
B_1	δ, δ	$4+3\delta, 4+3\delta$	$1+\delta, 5+\delta$
C_1	δ, δ	$5+\delta, 1+\delta$	$3+\delta, 3+\delta$

- In this game the best response for player i is:

$$BR_i(s_{-i}) = \begin{cases} A_i & \text{if } s_{-i} = A_{-i} \\ B_i & \text{if } s_{-i} = B_{-i} \ \& \ 4+3\delta \geq 5+\delta \\ C_i & \text{if } s_{-i} = B_{-i} \ \& \ 4+3\delta \leq 5+\delta \\ C_i & \text{if } s_{-i} = C_{-i} \end{cases}$$

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- (B_1, B_2) is a Nash equilibrium if $4+3\delta \geq 5+\delta$
- (B_1, B_2) is a Nash equilibrium if $\delta > 1/2$

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- (B_1, B_2) is a Nash equilibrium if $4+3\delta \geq 5+\delta$
- (B_1, B_2) is a Nash equilibrium if $\delta > 1/2$
- The strategy profile defined for Anna and Bob at the beginning of this section is indeed a subgame perfect Nash equilibrium if players value the future enough ($\delta > 1/2$)
- If players value the future enough ($\delta > 1/2$), then the future prize is worth the short term loss

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- In the repeated **Prisoner's Dilemma**, the stage game (played just once) had just one Nash equilibrium.
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- In the repeated **Prisoner's Dilemma**, the stage game (played just once) had just one Nash equilibrium.
- The only subgame perfect Nash equilibrium was to play the Nash equilibrium of the stage game in every period.
- In fact, one can prove generally that if **the stage game** has only one Nash equilibrium, then in the repeated game with that stage game, the unique subgame perfect Nash equilibrium requires the Nash equilibrium to be played in all periods and all information sets.

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- The only subgame perfect Nash equilibrium was to play the Nash equilibrium of the stage game in every period.
- In fact, one can prove generally that if **the stage game** has only one Nash equilibrium, then in the repeated game with that stage game, the unique subgame perfect Nash equilibrium requires the Nash equilibrium to be played in all periods and all information sets.
- In contrast, in this game, we saw that there was a subgame perfect Nash equilibrium in which an action profile (B_1, B_2) that was not a Nash equilibrium of the stage game was played in period 1.
- This was because there were **multiple** Nash equilibria of the stage game that could be used as rewards/punishments for certain behaviors.

- Are there any other action profiles that can be played in the first period?

Normal Form

	A_2	B_2	C_2
A_1	1, 1	0, 0	0, 0
B_1	0, 0	4, 4	1, 5
C_1	0, 0	5, 1	3, 3

- Are there any other action profiles that can be played in the first period?

Normal Form

	A_2	B_2	C_2
A_1	1, 1	0, 0	0, 0
B_1	0, 0	4, 4	1, 5
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- Suppose that the players were to play (A_1, C_2) in the first period.

- Are there any other action profiles that can be played in the first period?

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- Suppose that the players were to play (A_1, C_2) in the first period.
- Can this occur? The answer is no.

- Are there any other action profiles that can be played in the first period?

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	A_2	B_2	C_2
A_1	1, 1	0, 0	0, 0
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- Suppose that the players were to play (A_1, B_2) in the first period
- Can this occur? The answer is **no**
- Remember either (A_1, A_2) or (C_1, C_2) must be played in any pure strategy SPNE after a history

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- The maximum payoff that player 1 could get from playing according to this "supposed" SPNE:

$$\overset{\to 0}{u_1(A_1, B_2)} + \delta \overset{\to 0}{u_1(C_1, C_2)} = 3\delta$$

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$3\delta \geq 5 + \delta$
$2\delta > 5$
$\boxed{\delta \geq 5/2}$ No es posible
$\delta \in [0,1]$

- $5 + \delta$ is always greater than 3δ

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- Can there be a SPNE in which (A_1, C_2) is played in period 1?

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$$\overset{0}{u_1(A_1, C_2)} + \delta u_1(C_1, C_2) = 3\delta$$

- The worst payoff that player 1 can obtain by playing C_1 instead in period 1 is:

$$u_1(C_1, C_2) + \delta u_1(A_1, A_2) = 3 + \delta$$

- $3 + \delta$ is always greater than 3δ
- Thus, there are incentives to deviate

$3\delta \geq 3 + \delta$

$2\delta \geq 3$

$\boxed{\delta \geq 3/2}$ No es posible!

- Symmetrically there cannot be any SPNE in which (B_1, A_2) and (C_1, A_2) are played in period 1

- Symmetrically there cannot be any SPNE in which (B_1, A_2) and (C_1, A_2) are played in period 1
- We already know that $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ can be played in a SPNE in period 1

- Symmetrically there cannot be any SPNE in which (B_1, A_2) and (C_1, A_2) are played in period 1
- We already know that $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ can be played in a SPNE in period 1
- The remaining question is whether (C_1, B_2) can be played in period 1

- Consider the following strategy profile
- Player 1's strategy is:
  1. Play $\boxed{C_1}$ in period 1
  2. Play A_1 in period 2 if the first period action profile was (C_1, C_2)
  3. Play C_1 in period 2 if the first period action profile was anything other than (C_1, C_2)
- Player 2's strategy is:
  1. Play $\boxed{B_2}$ in period 1
  2. Play A_2 in period 2 if the first period action profile was (C_1, C_2)
  3. Play C_2 in period 2 if the first period action profile was anything other than (C_1, C_2)

¿EPS? $\boxed{t=2}$

$\boxed{T=1}$

$\boxed{J1}$ $U_1(C_1, B_2) + \delta U_1(C_1, C_2) \boxed{\geq} \begin{cases} U_1(A_1, B_2) + \delta U_1(C_1, C_2) \\ U_1(B_1, B_2) + \delta U_1(C_1, C_2) \end{cases}$

5 0 4

$\boxed{J2}$

$U_2(C_1, B_2) + \delta U_2(C_1, C_2) \geq \begin{cases} U_2(C_1, A_2) + \delta U_2(C_1, C_2) \\ \ldots \end{cases}$

- Player 1's strategy is:
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  1. Play B_2 in period 1
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$\boxed{S_2}$

$$U_2(C_1,B_2)+\delta U_2(C_1,C_2) \geq \begin{cases} U_2(C_1,A_2)+\delta U_2(C_1,C_2) \\ U_2(C_1,C_2)+\delta U_2(A_1,A_2) \end{cases}$$

$$1+\delta 3 \geq \begin{cases} \cdot\ 0+3\delta \\ \cdot\ 3+1\delta \end{cases}$$

$$1+3\delta \geq 3+\delta$$

$$2\delta \geq 2$$

$$\boxed{\delta \geq 1} \to \boxed{\delta = 1}$$

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- We know that the strategy is a NE in the subgames that start in $t = 2$
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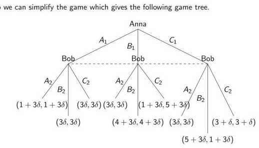
So we can simplify the game which gives the following game tree.



The normal form of this game (conditional on what happens in $T = 2$) is:

Normal Form

	A_2	B_2	C_2
A_1	$1+3\delta, 1+3\delta$	$3\delta, 3\delta$	$3\delta, 3\delta$
B_1	$3\delta, 3\delta$	$4+3\delta, 4+3\delta$	$1+3\delta, 5+3\delta$
C_1	$3\delta, 3\delta$	$5+3\delta, 1+3\delta$	$3+\delta, 3+\delta$

- In this game the best response for player i is:

$$BR_1(s_2) = \begin{cases} A_1 & \text{if } s_2 = A_2 \\ C_1 & \text{if } s_2 = B_2 \\ C_1 & \text{if } s_2 = C_2 \\ B_1 & \text{if } s_2 = C_2 \ \&\ \delta = 1 \end{cases}$$

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$$BR_2(s_1) = \begin{cases} A_2 & \text{if } s_1 = A_1 \\ C_2 & \text{if } s_1 = B_1 \\ C_2 & \text{if } s_1 = C_1 \\ B_2 & \text{if } s_1 = C_1 \ \&\ \delta = 1 \end{cases}$$

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- An equilibrium outcome of this game is to play (C_1, B_2) in period 1 and (C_1, C_2) in period 2 if $\delta = 1$

- There are other SPNE that results in the same equilibrium outcome
- For example consider the following SPNE
- Player 1's strategy is:
  1. Play C_1 in period 1.
  2. Play A_1 in period 2 if the first period action profile was anything other than (C_1, B_2).
  3. Play C_1 in period 2 if the first period action profile was (C_1, B_2).

  1. Player 2's strategy is:
  2. Play B_2 in period 1.
  3. Play A_2 in period 2 if the first period action profile was anything other than (C_1, B_2).
  4. Play C_2 in period 2 if the first period action profile was (C_1, B_2).

¿EPS? $\boxed{t=2}$ ✓

$\boxed{T=1}$ $$U_1(C_1,B_2)+\delta U_1(C_1,C_2) \geq \begin{cases} \cdot\ U_1(A,B_2)+\delta U_1(A_1,A) \\ \cdot\ U_1(B_1,B_2)+\delta U_1(A_1,A_2) \end{cases}$$

$$5+3\delta \geq \begin{cases} \cdot\ 0+\delta \\ \cdot\ 3+\delta \end{cases}$$

✓

$\boxed{S_2}$ $$U_2(C_1,B_2)+\delta U_2(C_1,C_2) \geq \begin{cases} \cdot\ U_2(C_1,A)+\delta U_2(A_1,A_2) \\ \cdot\ U_2(C_1,C_2)+\delta U_2(A_1,A_2) \end{cases}$$

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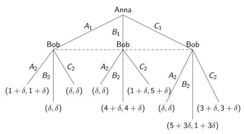

The normal form of this game (conditional on what happens in $T = 2$) is:

Normal Form

	A_2	B_2	C_2
A_1	$1+\delta, 1+\delta$	$3\delta, \delta$	δ, δ
B_1	δ, δ	$4+\delta, 4+\delta$	$1+\delta, 5+\delta$
C_1	δ, δ	$5+3\delta, 1+3\delta$	$3+\delta, 3+\delta$

- In this game the best response for player i is:

$$BR_1(s_2) = \begin{cases} A_1 & \text{if } s_2 = A_2 \\ C_1 & \text{if } s_2 = B_2 \\ C_1 & \text{if } s_2 = C_2 \end{cases}$$

- In this game the best response for player 2 is:

$$BR_2(s_1) = \begin{cases} A_2 & \text{if } s_1 = A_1 \\ C_2 & \text{if } s_1 = B_1 \\ C_2 & \text{if } s_1 = C_1 \\ B_2 & \text{if } s_1 = C_1 \text{ \& } \delta = 1 \end{cases}$$

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- Thus, characterizing all pure strategy SPNE is extremely tedious
- So instead of calculating all possible SPNE, lets just calculate the set of all possible equilibrium outcomes

$3+\delta$

$1+3\delta \geq 3+\delta$

$2\delta \geq 2$

$\boxed{\delta \geq 1} \rightarrow \boxed{\delta = 1}$

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T=1 T=2 cw T>1

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  1. $(A_1, A_2), (A_1, A_2)$
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  7. $(B_1, C_2), (C_1, C_2)$

$\cancel{(A_1, B_2)}$
$\cancel{(A_1, C_2)}$

- Can there be other equilibrium outcomes?

- We know that the following are possible equilibrium outcomes:
  1. [illegible]
  2. [illegible]
  3. [illegible]
  4. [illegible]
  5. [illegible]
  6. [illegible]
  7. [illegible]
- Can there be other equilibrium outcomes?

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  1. [illegible]
  2. [illegible]
  3. [illegible]
  4. [illegible]
  5. [illegible]
  6. [illegible]
  7. [illegible]
- Can there be other equilibrium outcomes? No.

- We know that the following are possible equilibrium outcomes:
  1. [illegible]
  2. [illegible]
  3. [illegible]
  4. [illegible]
  5. [illegible]
  6. [illegible]
  7. [illegible]
- Can there be other equilibrium outcomes? No. Why?

Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

Consider the following repeated game and $t = 1$

Stage Game

	A_2	B_2	C_2
A_1	(10, 10)	[illegible]	$(-1, 11)$
B_1	[illegible]	[illegible]	(0, 0)
C_1	[illegible]	[illegible]	[illegible]

$EN = \left\{ \begin{array}{l} (B_1, B_2) \\ (C_1, C_2) \end{array} \right\}.$

- The above game has two Nash equilibria (B_1, B_2) and (C_1, C_2)

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- Even though there are multiple Nash equilibria, there are no subgame perfect equilibria in which (A_1, A_2) is played in period 1

- The above game has two Nash equilibria (B_1, B_2) and (C_1, C_2)
- Even though there are multiple Nash equilibria, there are no subgame perfect equilibria in which (A_1, A_2) is played in period 1
- Either (B_1, B_2) or (C_1, C_2) must be played after deviating [illegible] since in the last period, always one of the stage game Nash equilibria must be played.

Case 1:

- Suppose that (B_1, B_2) is played in period 2 after (A_1, A_2) in period 1

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- Player 2 obtains a payoff of

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- Suppose that (B_1, B_2) is played in period 2 after (A_1, A_2) in period 1
- Player 2 obtains a payoff of

$$\boxed{10 + \delta}$$

- By deviating to B_2 in period 1, player 2 obtains at least:

$+3\delta$

$$11 + \delta$$

since in period 2 either (B_1, B_2) or (C_1, C_2) will be played in any SPNE

Case 1:

- Suppose that (B_1, B_2) is played in period 2 after (A_1, A_2) in period 1
- Player 2 obtains a payoff of

$$10 + \delta$$

- By deviating to B_2 in period 1, player 2 obtains at least:

$$11 + \delta$$

since in period 2 either (B_1, B_2) or (C_1, C_2) will be played in any SPNE

- Thus there are incentives to deviate

Case 2:

- Suppose instead that (C_1, C_2) is played in period 2 after (A_1, A_2) in period 1

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- Suppose instead that (C_1, C_2) is played in period 2 after (A_1, A_2) in period 1
- player 1 obtains a payoff of

$$10 + \delta$$

- By deviating to B_1 in period 1, player 1 obtains at least $11 + \delta$

$+3\delta$

Case 2:

- Suppose instead that (C_1, C_2) is played in period 2 after (A_1, A_2) in period 1
- player 1 obtains a payoff of

$$10 + \delta$$

- By deviating to B_1 in period 1, player 1 obtains at least $11 + \delta$
- Thus there are incentives to deviate

- Even though there are multiple NE in the stage game, it may still be impossible to achieve Pareto efficient action profiles in period 1

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- The key to this example was that players disagreed on which stage game NE is better.

- Even though there are multiple NE in the stage game, it may still be impossible to achieve Pareto efficient action profile in period 1.

- The key to this example was that players disagreed on which stage game NE is better.

- Thus, at least one of the two players had an incentive to deviate away from (A_1, A_2) in period 1.

Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

Lecture 18: Repeated Games

Recap from last class

More than one NE in the stage game

Example 1

Example 2

- Even if there is disagreement about which stage game NE is better between the two players, we can still obtain examples of outcomes that are not Nash equilibrium in the first period.

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- Consider for example the following stage game and suppose we consider a twice repeated game with discount factor δ.

Stage Game

$$\text{EN}\left\{\begin{matrix}(B_1, B_2)\\ (C_1, C_2)\end{matrix}\right\}$$

- The NE of the stage game are (B_1, B_2) and (C_1, C_2).

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- In this repeated game, is there a subgame perfect Nash equilibrium in which (A_1, A_2) is played in period 1?

- The answer is yes.

- Consider the following strategy profile.

- Player 1 plays the following strategy:
  1. A_1 in period 1
  2. B_1 in period 2 if (A_1, A_2) was played in period 1;
  3. C_1 in period 2 if (A_1, A_2) was not played in period 1.

- Player 2 plays the following strategy:
  1. A_2 in period 1
  2. B_2 in period 2 if (A_1, A_2) was played in period 1;
  3. C_2 in period 2 if (A_1, A_2) was not played in period 1.

¿EPS? $\boxed{t=2}$ ✓

$\boxed{T=1}$ $\boxed{J_1}$ $U_1(A_1, A_2) + \delta U_1(B_1, B_2) \geq \begin{cases} \cdot\ U_1(B_1, A_2) + \delta U_1(C_1, C_2) \\ \cdot\ U_1(C_1, A_2) + \delta U_1(C_1, C_2) \end{cases}$

$10 + 3\delta \geq 1 \cdot 11 + \delta$

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- Player 2 plays the following strategy:
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- no (if $\delta < \frac{1}{2}$)!

Stage Game

	A_2	B_2	C_2
A_1	$(10, 10)$	$(0, 9)$	$(0, 9)$
B_1	$(11, -1)$	$(3, 1)$	$(0, 0)$
C_1	$(11, -2)$	$(0, 0)$	$(1, 3)$

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	A_2	B_2	C_2
A_1	$(10, 10)$	$(0, 9)$	$(0, 9)$
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- Player 1:
- If he follows: $u_1 = 10 + 3\delta$
- If he defects: $u_1 = 11 + \delta$
- Follows if $\delta \geq \frac{1}{2}$

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• $U_1(C_1, A_2) + \delta U_1(C_1, C_2)$

$$10 + 3\delta \geq \begin{cases} \cdot\ 11 + \delta \\ \cdot\ 11 + \delta \end{cases}$$

$$10 + 3\delta \geq 11 + \delta$$

$$2\delta \geq 1$$

$$\boxed{\delta \geq 1/2}$$

$$\boxed{J_2}\ U_2(A_1, A_2) + \delta U_2(B_1, B_2) \geq \begin{cases} \cdot\ U_2(A_1, B_2) + \delta U_2(C_1, C_2) \\ \cdot\ U_2(A_1, C_2) + \delta U_2(C_1, C_2) \end{cases}$$

$$10 + \delta \geq \begin{cases} \cdot\ 9 + 3\delta \\ \cdot\ 9 + 3\delta \end{cases}$$

$$10 + \delta \geq 9 + 3\delta$$

$$1 \geq 2\delta$$

$$\boxed{\delta \leq 1/2}$$

$$\boxed{\delta = 1/2}$$

- Player 2:
- If he follows: $u_2 = 10 + \delta$
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- Player 2:
- If he follows: $u_2 = 10 + \delta$
- If he defects: $u_2 = 9 + 3\delta$
- Follows if $\delta \leq \frac{1}{2}$
- Can only be a SPNE is $\delta = \frac{1}{2}$

- The key here is that player 2 by breaking the agreement in period 1 moves the period 2 play to his favored stage game NE of (C_1, C_2)

- Suppose we flipped the roles of B and C and considered the following strategy profile
- Player 1 plays the following strategy:
  1. A_1 in period 1;
  2. C_1 in period 2 if (A_1, A_2) was played in period 1;
  3. B_1 in period 2 if (A_1, A_2) was not played in period 1.
- Player 2 plays the following strategy:
  1. A_2 in period 1;
  2. C_2 in period 2 if (A_1, A_2) was played in period 1;
  3. B_2 in period 2 if (A_1, A_2) was not played in period 1.

- This is not a SPNE either because now player 1 has a definitive incentive to deviate from (A_1, A_2) in period 1

Stage Game

	A_2	B_2	C_2
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- Player 1:
  - If he follows: $u_1 = 10 + \delta$
  - If he defects: $u_1 = 11 + 3\delta$
  - Always defects

- So how do we construct a SPNE with (A_1, A_2) played in period 1?

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- This is because in period 1 player 2 is best responding **myopically** at (A_1, A_2) already

- So how do we construct a SPNE with (A_1, A_2) played in period 1?
- The key here is to notice that player 2 does not need to be punished in period 2 from breaking the agreement in period 1
- This is because in period 1 player 2 is best responding **myopically** at (A_1, A_2) already
- In other words, need to be punished **only if** the player has a deviation that benefits him **myopically** or in the short term

- Player 1 plays the following strategy:
  1. A_1 in period 1;
  2. B_1 in period 2 if player 1 played A_1;
  3. C_1 in period 2 if player 1 played B_1 or C_1.
- Player 2 plays the following strategy:
  1. A_2 in period 1;
  2. B_2 in period 2 if player 1 played A_1;
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	A_2	B_2	C_2
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  - If he follows: $u_1 = 10 + 3\delta$
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- Player 2:
  - If he follows: $u_2 = 10 + X\delta$

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  - Follows if $\delta \geq \frac{1}{2}$
- Player 2:
  - If he follows: $u_2 = 10 + X\delta$
  - If he defects: $u_2 = 9 + X\delta$

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  - Follows