Lecture 19: Infinitely Repeated Games

Mauricio Romero

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Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games

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When the game is instead infinitely repeated, this argument no longer applies since there is no such thing as a last period

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Then play moves to period t + 1 and the game continues in the same manner.

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▶ We can represent each information set of player *i* by a history:

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We denote the set of all histories at time t as H^t

Prisoner's Dilemma

	<i>C</i> ₂	D_2	
C_1	1,1	-1, 2	
D_1	2, -1	0,0	

 $\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$

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- For time t, H^t consists of 4^t possible histories
- This means that there is a one-to-one mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree
- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player *i* in each time *t*

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Therefore, it is a function that describes:

$$s_i: \bigcup_{t\geq 0} H^t o A_i.$$

Intuitively, s_i describes exactly what player i would do at every possible history h^t, where s_i(h^t) describes what player i would do at history h^t

For example in the infinitely repeated prisoner's dilemma, the strategy s_i(h^t) = C_i for all h^t and all t is the strategy in which player i always plays C_i regardless of the history

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There can be more complicated strategies such as the following:

$$s_i(h^t) = egin{cases} C_i & ext{ if } t = 0 ext{ or } h^t = (C, C, \dots, C), \ D_i & ext{ otherwise.} \end{cases}$$

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The above is called a grim trigger strategy

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lntuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

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Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain Lets see some examples of how to compute payoffs in the repeated game

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- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again (C, C, ...)
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

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Then if both players play these strategies, then the sequence of actions that arise is:

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Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta-1}{1-\delta^2}.$$

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$$(a^0, a^1, \ldots, a^{t-1}, a^t(s_i, s_{-i} \mid h^t), a^{t+1}(s_i, s_{-i} \mid h^t) \ldots),$$

where $a^{\tau}(s_i, s_{-i} \mid h^t)$ denotes the action profile that will be played at time $\tau \geq t$ if players indeed play the strategy profile (s_i, s_{-i}) after history h^t

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Then we can define the following payoff to the strategy profile (s_i, s_{-i}) conditional on the history h^t:

$$U_i(s_i, s_{-i} \mid h^t) = \sum_{\tau=0}^{t-1} \delta^{\tau} u_i(a^{\tau}) + \sum_{\tau=t}^{\infty} \delta^{\tau} u_i(a^{\tau}(s_i, s_{-i} \mid h^t)).$$

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- The value W_i(s_i, s_{-i} | h^t) represents the value that i accrues in this subgame, following history h^t, when players play according to h^t, viewing payoffs from time t perspective (as if time t is time 0)

▶ We can represent the payoff $U_i(s_i, s_{-i} | h^t)$ using continuation values:

$$U_i(s_i, s_{-i} \mid h^t) = \sum_{\tau=0}^{t-1} \delta^{\tau} u_i(a^{\tau}) + \delta^t W_i(s_i, s_{-i} \mid h^t).$$

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• $W_i(s_i, s_{-i} \mid h^t)$ can also be decomposed as follows:

$$W_i(s_i, s_{-i} \mid h^t) = u_i(s_i(h^t), s_{-i}(h^t)) + \delta W_i(s_i, s_{-i} \mid (h^t, s_i(h^t), s_{-i}(h^t)))$$

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games Subgame Perfect Nash Equilibrium Examples

What is a subgame perfect Nash equilibrium in an infinitely repeated game?

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That is a strategy profile s = (s₁,..., s_n) is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

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- First notice that a particular subgame corresponds to an infinitely repeated game that starts after a certain history h^t
- ► Furthermore the fact that s is a Nash equilibrium after the history means that after every history h^t = (a⁰,..., a^{t-1}), s_i is a best response against s_{-i} at such a history:

$$U_i(s_i, s_{-i} \mid h^t) = \max_{s'_i} U_i(s'_i, s_{-i} \mid h^t).$$

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$$U_i(s_i, s_{-i} \mid h^t) = \max_{s'_i} U_i(s'_i, s_{-i} \mid h^t).$$

• Rewriting the above we get that for all s'_i ,

$$\sum_{\tau=0}^{t-1} \delta^{\tau} u_i(a^{\tau}) + \delta^t W_i(s_i, s_{-i} \mid h^t)$$

$$\geq \sum_{\tau=0}^{t-1} \delta^{\tau} u_i(a^{\tau}) + \delta^t W_i(s'_i, s_{-i} \mid h^t)$$

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Furthermore we can divide all utilities by δ^t to realize that:

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The above is still a bit complicated since checking that a strategy s_i is a best response against s_{-i} may be quite difficult since there are infinitely many pure strategies s'_i that player i could potentially deviate to

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► However, the following proposition makes the check quite simple

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if for all times t, each history h^t, and each player i,

$$u_i(s_i(h^t), s_{-i}(h^t)) + \delta W_i(s_i, s_{-i} \mid (h^t, s_i(h^t), s_{-i}(h^t))) \\ = \max_{a_i^t \in A_i} u_i(a_i^t, s_{-i}(h^t)) + \delta W_i(s_i, s_{-i} \mid (h^t, a_i^t, s_{-i}(h^t))).$$

In words the above states that if s is a subgame perfect Nash equilibrium if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s' from time t + 1 on

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This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s'_i.

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We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games Subgame Perfect Nash Equilibrium Examples

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Lets go back to the infinitely repeated prisoner's dilemma

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► Why is this a SPNE?

We can use the one-stage deviation principle

Prisoner's Dilemma

	<i>C</i> ₂	D_2	
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• Under this strategy profile s_1^*, s_2^* , for all histories h^t ,

$$W_1(s_1^*, s_2^* \mid h^t) = W_2(s_1^*, s_2^* \mid h^t) = 0.$$

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> Thus, for all histories h^t ,

$$\underbrace{u_i(D_i, D_{-i})}_0 + \delta \underbrace{W_i(s_1^*, s_2^* \mid h^t)}_0 > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{W_i(s_1^*, s_2^* \mid h^t)}_0$$

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• Thus,
$$(s_1^*, s_2^*)$$
 is a SPNE

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In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^{*} be a Nash equilibrium of the stage game. Then the strategy profile s^{*} in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

▶ We use the one-stage deviation principle again

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- We need to show that for every t and all h^t , and all i,

$$u_i(s_i^*(h^t), s_{-i}^*(h^t)) + \delta W_i(s_i^*, s_{-i}^* \mid (h^t, s_i^*(h^t), s_{-i}^*(h^t))) \\ = \max_{a_i' \in A_i} u_i(a_i', s_{-i}^*(h^t)) + \delta W_i(s_i^*, s_{-i}^* \mid (h^t, a_i', s_{-i}^*(h^t))).$$

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When the repeated game is infinitely repeated, this is no longer true

Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{ if } h^t = (C, C, \dots, C) \\ D_i & \text{ if } h^t \neq (C, C, \dots, C). \end{cases}$$

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▶ The equilibrium path of play for this SPNE is for players to play C in every period

How do we show that the above is indeed an SPNE?

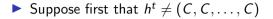
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• We need to check the one-stage deviation principle at every history h^t .



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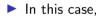
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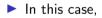
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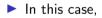
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▶ Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \ge 1/2$.

The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^{*} is a Nash equilibrium of the stage game. Suppose that â is an action profile of the Nash equilibrium such that

$$u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).$$

Then there is some $\delta^* < 1$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play \hat{a} in every period.

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 concludes the proof

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