Lecture 19

Thursday, April 29, 2021 2:21 PM

Lecture19...

$(G, \boxed{T}) \rightarrow$ FINITO

$\rightarrow 1$ EN $\rightarrow$ EPS $\Rightarrow$ EN $\forall$E.

$+1$ EN $\rightarrow$ VARIOS CASOS.

Lecture 19: Infinitely Repeated Games

Mauricio Romero

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Introduction to Infinitely Repeated Games

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- This happened because there was a last period from which we could induct backwards (and there was a domino effect!)
- When the game is instead **infinitely** repeated, this argument no longer applies since there is no such thing as a last period

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- Then play moves to period $t + 1$ and the game continues in the same manner.

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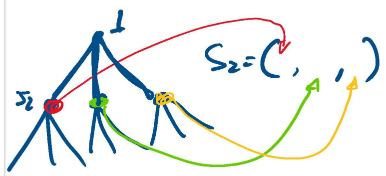
- We denote the set of all histories at time t as H^t

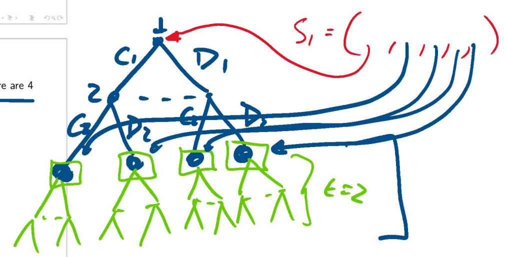


Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$\boxed{0, 0}$

- For example, if the stage game is the prisoner's dilemma, at period 1, there are 4 possible histories:

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$$



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- For time t, H^t consists of 4^t possible histories
- This means that there is a **one-to-one** mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree
- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player i in each time t

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- Intuitively, s_i describes exactly what player i would do at every possible history h^t, where $s_i(h^t)$ describes what player i would do at history h^t

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- There can be more complicated strategies such as the following:

$$s_i(h^t) = \begin{cases} C_i & \text{if } t = 0 \text{ or } h^t = (C, C, \ldots, C), \\ D_i & \text{otherwise.} \end{cases}$$

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- The above is called a **grim trigger strategy**

→ OJO-POR-OJO
TIT-FOR-TAT

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- However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.
- Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain

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- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \ldots)$

$$\boxed{\begin{array}{l} s_1(h^t) = C_1 \\ s_2(h^t) = C_2 \end{array}}$$

$$U_1(s_1, s_2) = U_1(C_1, C_2)\delta^0 + U_1(C_1, C_2)\delta^1 + U_1(C_1, C_2)\delta^2 + \ldots$$

$$= U_1(C_1, C_2)\left[\delta^0 + \delta^1 + \delta^2 + \delta^3 + \ldots\right]$$

$$= U_1(C_1, C_2)\boxed{\sum_{t=0}^{\infty} \delta^t}$$

$$\boxed{= \underbrace{U_1(C_1, C_2)}_{1}\frac{1}{1-\delta}}$$

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$$\sum_{t=0}^{T} \delta^t = \boxed{1} + \delta + \cancel{\delta^2} + \ldots + \cancel{\delta^T}$$

$$-\delta\sum_{t=0}^{T} \delta^t = -\cancel{\delta} - \cancel{\delta^2} - \cancel{\delta^3} - \ldots - \delta^{T+1}$$

$$\sum_{t=0}^{T} \delta^t - \delta\sum_{t=0}^{T} \delta^t = 1 - \delta^{T+1}$$

$$\sum_{t=0}^{T}(1-\delta) = 1 - \delta^{T+1}$$

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- Thus the payoffs of all players is again $\boxed{\frac{1}{1-\delta}}$

$$\overset{t=1}{(C_1, C_2)}, \overset{t=2}{(C_1, C_2)}, \overset{t=3}{(C_1, C_2)}, \ldots$$

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- Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta - 1}{1-\delta^2}.$$

Caso Por Caso

$$U_1(C_1,D_2) + \delta U_1(D_1,C_2) + \delta^2 U(C_1,D_2) + \delta^3 U(D_1,C_2) \dots$$

$$U_1(C_1,D_2) \boxed{\sum_{t=0}^{\infty} \delta^{2t}} + U(D_1,C_2) \boxed{\sum_{t=0}^{\infty} \delta^{2t+1}}$$

$$U_1(C_1,D_2) \frac{1}{1-\delta^2} + U(D_1,C_2) \frac{\delta}{1-\delta^2}$$

$$\delta \sum_{t=0}^{\infty} \delta^{2t}$$

$$\delta \sum_{t=0}^{\infty} (\delta^2)^t$$

$$\boxed{\delta \frac{1}{1-\delta^2}}$$

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games
Subgame Perfect Nash Equilibrium
Examples

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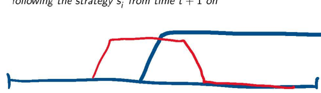
- It is exactly the same idea as in the finitely repeated game or more generally extensive form games
- That is a strategy profile $s = (s_1, \dots, s_n)$ is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

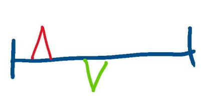
$\boxed{S}$ en $(6, \infty)$

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s_i' from time $t+1$ on



s



$\boxed{S} \in N \ (6, \infty)$

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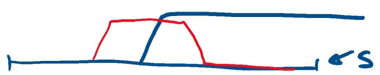

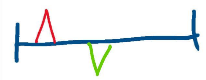

- This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s_i'.

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- We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

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- What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D_1 and D_2 always at all information sets
- Why is this a SPNE?

- We can use the one-stage deviation principle

	C_2	D_2
C_1		
D_2		

Prisoner's Dilemma

	C_2	D_2
C_1	1, 1	−1, 2
D_1	2, −1	0, 0

$EN = (D_1, D_2)$

EPS → JUGAR EN $\forall t$

$$V(s_t) = \sum_{t=0}^{\infty} \delta^t 0 = 0$$

$$V(\text{Desvio}) = -1 + \sum_{t=1}^{\infty} \delta^t 0 = -1$$

COMPARACION DESV EN $t=0$

PERIODO T

$$V(\text{No Desv}) = \cancel{V^{ND}_{pasado}} + V^{ND}_T + V^{ND}_{futuro} = V_{pasado} + 0 + \sum_{t=1}^{\infty} \delta^t 0 = \boxed{V_{pasado}}$$

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$$\boxed{V^{ND}_{pasado} = V^{D}_{pasado}}$$

- Under this strategy profile s_1^*, s_2^*, for all histories h^t,
$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

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- Thus, for all histories h^t,
$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

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In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^* be a Nash equilibrium of the stage game. Then the strategy profile s^* in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

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- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium

- When the repeated game is infinitely repeated, this is no longer true

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- Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \ldots, C) \\ D_i & \text{if } h^t \neq (C, C, \ldots, C). \end{cases}$$

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- The *equilibrium path of play* for this SPNE is for players to play C in every period → Si $\boxed{\delta \geq 1/2}$

→ RESULTADO ASOCIADO

- How do we show that the above is indeed an SPNE? → OBJETIVO: NO HAY INCENTIVOS A DESV.

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- We need to check the one-stage deviation principle at every history h^t.

Case 1:

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- Players are each suppose to play D_i

$\boxed{h^t \neq (C, C, \dots, C)}$ CASO #1

$$V_{DESV} = U_i(C_i, D_{-i}) + \sum_{t=1}^{\infty} U_i(D_i, D_{-i})\delta^t = -1 + U(D_i, D_{-i}) \boxed{\sum_{t=1}^{\infty} \delta^t} = -1 + U(D_i, D_{-i}) \frac{\delta}{1-\delta} = -1 + 0 \cdot \frac{\delta}{1-\delta} = -1$$

$$V_{ND} = \sum_{t=0}^{\infty} U_i(D_i, D_{-i})\delta^t = U_i(D_i, D_{-i}) \boxed{\sum_{t=0}^{\infty} \delta^t} = U(D_i, D_{-i}) \frac{1}{1-\delta} = 0 \cdot \frac{1}{1-\delta} = 0$$

$h^t = (C, C, \dots, C)$ CASO #2

$$V_{ND} = \sum_{\boxed{t=0}}^{\infty} U_i(C_i, C_{-i})\delta^t = U_i(C_i, C_{-i}) \sum_{t=0} \delta^t = U_i(C_i, C_{-i}) \frac{1}{1-\delta} = \frac{1}{1-\delta}$$

$$V_{DESV} = \underbrace{U_i(D_i, C_{-i})}_{\to\, t+1:\ h^t \neq (C, \dots, C)} + \sum_{t=1}^{\infty} U_i(D_i, D_{-i})\delta^t = 2 + U_i(D_i, D_{-i}) \sum_{t=1}^{\infty} \delta^t = 2 + \cancelto{0}{U_i(D_i, D_{-i})} \frac{\delta}{1-\delta} = 2$$

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- But this is satisfied since D is a Nash equilibrium of the stage game

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$\hookrightarrow T+1$

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$$V_{ND} \geq V_{DESV}$$

$$\frac{1}{1-\delta} \geq 2$$

$$1 \geq 2 - 2\delta$$

$$2\delta \geq 1$$

$$\boxed{\delta \geq \frac{1}{2}}$$

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- Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \geq 1/2$.

- The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^* is a Nash equilibrium of the stage game. Suppose that $\hat{a}$ is an action profile of the Nash equilibrium such that

$$\boxed{u_1(\hat{a}) > u_1(a^*), \dots, u_n(\hat{a}) > u_n(a^*).}$$

Then there is some $\delta^* < 1$ such that whenever $\delta \geq \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play $\hat{a}$ in every period.

1) NO DICE CUANTO ES δ^*

2) NO NOS DICE δ^* Q' SOSTIENE JUGAR $\hat{a}$ $\forall t$