Lecture 19

Thursday, April 29, 2021 2:21 PM

Lecture19...

$(G, T) \to \boxed{\text{FINITO}}$

$\to$ 1 EN $\to$ EPS EN $\forall t$

$\to$ +1 EN $\to$ PUEDE PASAR "DE TODO"

Lecture 19: Infinitely Repeated Games

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Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games

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- One of the features of **finitely** repeated games was that if the stage game had a **unique** Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium
- This happened because there was a last period from which we could induct backwards (and there was a domino effect!)
- When the game is instead **infinitely** repeated, this argument no longer applies since there is no such thing as a last period

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- Then play moves to period $t + 1$ and the game continues in the same manner.

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- We can represent each information set of player i by a history:

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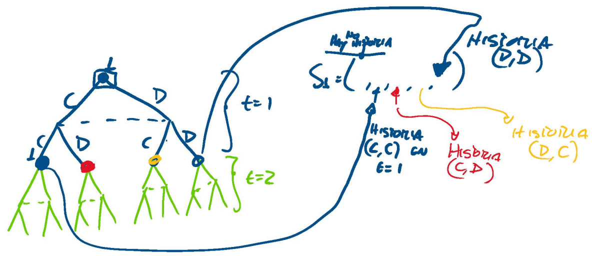
- We denote the set of all histories at time t as H^t



Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$0, 0$

- For example, if the stage game is the prisoner's dilemma, at period 1, there are 4 possible histories:

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$$

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- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player i in each time t

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- Therefore, it is a function that describes:

$$\boxed{s_i : \bigcup_{t \geq 0} H^t \to A_i.}$$

- Intuitively, s_i describes exactly what player i would do at every possible history h^t, where $s_i(h^t)$ describes what player i would do at history h^t

$s_i(h^t)$

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- There can be more complicated strategies such as the following:

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- The above is called a **grim trigger strategy** → GATillo

TIT-FOR-TAT
OJO-POR-OJO.

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$\delta \rightarrow$ PROB MAÑAN ENFRENTARNOS Y JUGAR G

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- Suppose the *strategies* $s_1, \dots, s_n$ are played which lead to the infinite sequence of action profiles:
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- Then the payoff of player i in this repeated game is given by:
$$\boxed{\sum_{t=0}^{\infty} \delta^t u_i(a^t).} = U_{t=0}\delta^0 + U_{t=1}\cdot\delta^1 + U_{t=2}\delta^2 + \dots +$$
- Intuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

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- However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.
- Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain

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$$U_i(C_1,C_2) \cdot \left[1 + \delta + \delta^2 + \dots\right]$$

$$U_i(C_1,C_2) \boxed{\sum_{t=0}^{\infty} \delta^t}$$

$$U_i(C_1,C_2) \cdot \frac{1}{1-\delta}$$

$$\boxed{\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}}$$

$$\sum_{t=0}^{T} \delta^t = 1 + \cancel{\delta} + \cancel{\delta^2} + \dots + \cancel{\delta^T}$$

$$+ \quad -\delta \sum_{t=0}^{T} \delta^t = -\cancel{\delta} - \cancel{\delta^2} - \cancel{\delta^3} - \dots - \cancel{\delta^T} - \delta^{T+1}$$

$$\sum_{t=0}^{T} \delta^t - \delta \sum_{t=0}^{T} \delta^t = 1 - \delta^{T+1}$$

$$\left(\sum_{t=0}^{T} \delta^t\right)(1-\delta) = 1 - \delta^{T+1}$$

$$\sum_{t=0}^{T} \delta^t = \frac{1-\delta^{T+1}}{1-\delta}$$

$$\lim_{T\to\infty} \sum_{t=0}^{T} \delta^t = \lim_{T\to\infty} \frac{1-\cancel{\delta^{T+1}}^{\to 0}}{1-\delta} \to \frac{1}{1-\delta}$$

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- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all $i = 1, 2$ and all h^t.
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- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \dots)$

$$\overset{T=1}{(C_1,C_2)},\ \overset{T=2}{(C_1,C_2)},\ \overset{T=3}{(C_1,C_2)},\ \dots$$

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- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all $i = 1, 2$ and all h^t.
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- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \dots)$
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

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- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period

OJO POR OJO

$$U_1(C_1,D_2) + \delta U_1(D_1,C_2) + \delta^2 U_1(C_1,D_2) + \delta^3 U_1(D_1,C_2) + \dots$$

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$$U_1(C_1,D_2)\left[1+\delta^2+\delta^4+\dots\right] + U_1(D_1,C_2)\left[\delta+\delta^3+\delta^5+\dots\right]$$

$$U_1(C_1,D_2)\boxed{\sum_{t=0}^{\infty}\delta^{2t}} + U_1(D_1,C_2)\boxed{\sum_{t=0}^{\infty}\delta^{2t+1}} \longrightarrow \delta\sum_{t=0}^{\infty}\delta^{2t} = \delta\left(\sum_{t=0}^{\infty}(\delta^2)^t\right) = \frac{\delta}{1-\delta^2}$$

$$\boxed{U_1(C_1,D_2)\frac{1}{1-\delta^2} + U_1(D_1,C_2)\frac{\delta}{1-\delta^2}}$$

- How about a more complicated strategy profile?
- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
- Then if both players play these strategies, then the sequence of actions that arise is:

$$(C_1, D_2), (D_1, C_2), (C_1, D_2), \dots$$

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- Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty}\delta^{2t}(-1) + \delta^{2t+1}\cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta-1}{1-\delta^2}.$$

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games
Subgame Perfect Nash Equilibrium
Examples

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- It is exactly the same idea as in the finitely repeated game or more generally extensive form games
- That is a strategy profile $s = (s_1, \dots, s_n)$ is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s_i' from time $t+1$ on

oso-Por-oso

- This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s_i'.

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- We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

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- Lets go back to the infinitely repeated prisoner's dilemma
- What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D_1 and D_2 always at all information sets
- Why is this a SPNE?
- We can use the one-stage deviation principle

Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
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EN = (D_1, D_2)

$(6, \infty)$

- Under this strategy profile s_1^*, s_2^*, for all histories h^t,
$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

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- Thus, for all histories h^t,
$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

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- Thus, for all histories h^t,
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- Thus, (s_1^*, s_2^*) is a SPNE

In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^* be a Nash equilibrium of the stage game. Then the strategy profile s^* in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

$\boxed{S_t}$

$\boxed{\text{JUGAR } (D_1, D_2) \ \forall t}$

$$V_{ND} = \cancel{V_{ND}^{PASADO}} + V_{ND}^{PRESENTE} + V_{ND}^{FUTURO} = U(D_1, D_2) + \boxed{\sum_{t=1}^{\infty} U(D_1, D_2)\delta^t} = 0 + \cancel{U(D_1, D_2)}\sum_{t=1}^{\infty}\delta^t = 0$$

$$V_D = \cancel{V_D^{PASADO}} + V_D^{PRES.} + V_D^{FUTURO} = U(C_1, D_2) + \boxed{\sum_{t=1}^{\infty} U(D_1, D_2)\delta^t} = -1 + \cancel{U(D_1, D_2)}\sum_{t=1}^{\infty}\delta^t = -1$$

$V_{ND}^{PASADO} = V_D^{PASADO}$ (YO NO PUEDO ALTERAR EL PASADO)

$\boxed{V_{ND} > V_D}$ ✓

JUGAR (D_1, D_2) $\forall t$ ES UN EPS

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- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium
- When the repeated game is infinitely repeated, this is no longer true

- Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \ldots, C) \\ D_i & \text{if } h^t \neq (C, C, \ldots, C). \end{cases}$$

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- We will show that if δ is sufficiently high, so that the players are sufficiently patient, the strategy profile of grim trigger strategies is indeed a SPNE
- The *equilibrium path of play* for this SPNE is for players to play C in every period

- How do we show that the above is indeed an SPNE?

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$\boxed{\text{EPS}}$?

CASO # 1 $\rightarrow \boxed{h_t \neq (C, \ldots, C)}$

$$V_D = U_i(C_i, D_{-i}) + \sum_{t=1}^{\infty} U_i(D_i, D_{-i})\delta^t = U_i(C_i, D_{-i}) + U(D_i, D_{-i})\boxed{\sum_{t=1}^{\infty}\delta^t} = U_i(C_i, D_{-i}) + \cancel{U(D_i, D_{-i})}^{\to 0}\frac{\delta}{1-\delta} = -1$$

$$V_{ND} = \sum_{t=0}^{\infty} U_i(D_i, D_{-i})\delta^t = U_i(D_i, D_{-i})\boxed{\sum_{t=0}^{\infty}\delta^t} = \cancel{U(D_i, D_{-i})}^{\to 0} \cdot \frac{1}{1-\delta} = 0$$

$$\boxed{V_{ND} > V_D}$$

$$\boxed{\text{CASO } \# 2 \quad h_t = (C, \ldots, C)}$$

$$V_{ND} = \sum_{t=0}^{\infty} U_i(C_i, C_{-i})\delta^t = U_i(C_i, C_{-i})\sum_{t=0}^{\infty}\delta^t = U(C_i, C_{-i})\frac{1}{1-\delta} = 1 \cdot \frac{1}{1-\delta}$$

$$V_D = U_i(D_i, C_{-i}) + \sum_{t=1}^{\infty} U_i(D_i, D_{-i})\delta^t = U_i(D_i, C_{-i}) + U_i(D_i, D_{-i})\sum_{\boxed{t=1}}^{\infty}\delta^t$$

$\hookrightarrow h_{t+1} \neq (C, \ldots, C)$

$$= U_i(D_i, C_{-i}) + U_i(D_i, D_{-i})\frac{\delta}{\ldots}$$

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- How do we show that the above is indeed an SPNE?
- We use the one-stage deviation principle again
- We need to check the one-stage deviation principle at every history h^t.

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$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\ \geq u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

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- But since $h^t \neq (C, C, \dots, C)$, $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.

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- But since $h^t \neq (C, C, \dots, C)$, $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.
- So the above inequality is satisfied if and only if

$$u_i(D_i, D_{-i}) \geq u_i(C_i, D_{-i}).$$

$\omega h_{t+1} \neq (C, \dots, C)$

$= V_i(D_i, C_{-i}) + V_i(D_i, D_{-i}) \frac{\delta}{1-\delta}$

$= 2 + 0 \cdot \frac{\delta}{1-\delta} = 2$

$V_{ND} \geq V_D$

$\frac{1}{1-\delta} \geq 2$

$1 \geq 2 - 2\delta$

$2\delta \geq 1$

$\boxed{\delta \geq \frac{1}{2}}$

Case 1:

- Suppose first that $h^t \neq (C, C, \dots, C)$
- Players are each suppose to play D_i
- Thus, we need to check that

$$
\begin{aligned}
&u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\
&\geq u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))
\end{aligned}
$$

- But since $h^t \neq (C, C, \dots, C)$,
  $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.
- So the above inequality is satisfied if and only if

$$u_i(D_i, D_{-i}) \geq u_i(C_i, D_{-i}).$$

- But this is satisfied since D is a Nash equilibrium of the stage game

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- Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \geq 1/2$.

- The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^* is a Nash equilibrium of the stage game. Suppose that $\hat{a}$ is an action profile of the Nash equilibrium such that

$$u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).$$

$U_i(\hat{a}) > U_i(a^*) \ \forall i$

Then there is some $\boxed{\delta^*} < 1$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play $\hat{a}$ in every period.

↳ 1) NO DICE CUANTO δ^*

2) NO NOS DICE CUAL δ^* ES