Lecture 2: General Equilibrium

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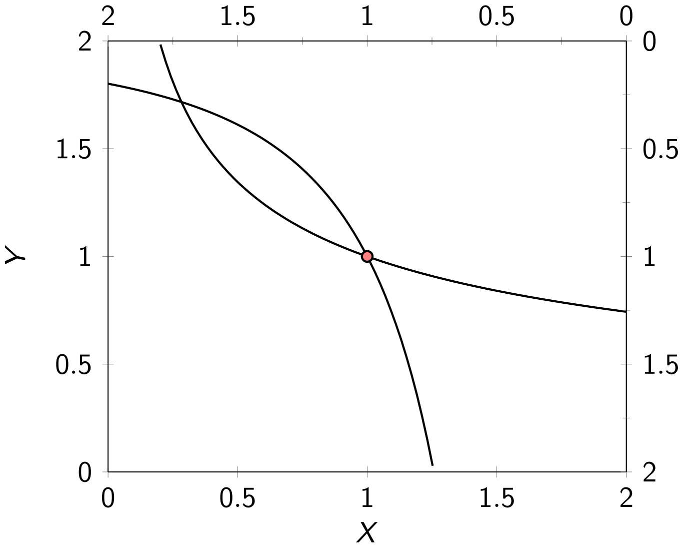
Cobb-Douglas

$$u_A(x, y) = x^{\alpha} y^{1-\alpha}$$
$$u_B(x, y) = x^{\beta} y^{1-\beta}$$

For graph suppose

$$\alpha = 0.7$$
$$\beta = 0.3$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

Cobb-Douglas



Cobb-Douglas

- Indifference curves must be tangent (formalize this later)
- Thus, the MRS must be equalized across the two consumers

$$MRS^A_{x,y} = \frac{\frac{\partial x^\alpha y^{1-\alpha}}{\partial x}}{\frac{\partial x^\alpha y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha} \frac{x^{\alpha-1} y^{1-\alpha}}{x^\alpha y^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{y^A}{x^A}$$

$$MRS^B_{x,y} = \frac{\frac{\partial x^\beta y^{1-\beta}}{\partial x}}{\frac{\partial x^\beta y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta} \frac{x^{\beta-1} y^{1-\beta}}{x^\beta y^{-\beta}} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

$$\frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

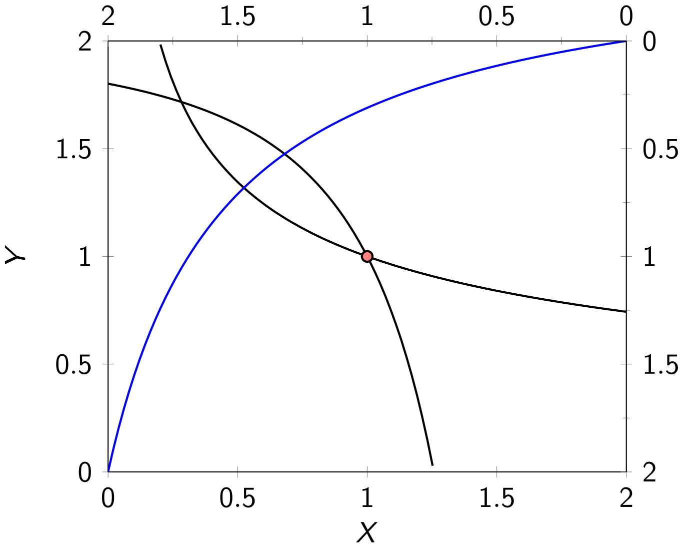
$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Then:

$$y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}$$

Cobb-Douglas



Lecture 2: General Equilibrium

Using calculus

Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$

$$\begin{aligned} u_B(x^B, y^B) &\geq \underline{u}_B = u_B(x^{B*}, y^{B*}) \\ x^B + x^A &\leq \omega_x, \\ y^B + y^A &\leq \omega_y. \end{aligned}$$

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ *is Pareto efficient if and only if it solves*

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$

$$\begin{aligned} u_B(x^B, y^B) &\geq \underline{u}_B \\ x^B + x^A &\leq \omega_x, \\ y^B + y^A &\leq \omega_y. \end{aligned}$$

- Very tempting to use lagrangeans, no?

- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\mathcal{L} = u_A(x^A, y^A) + \lambda(u_B(\omega_x - x^A, \omega_y - x^B) - \underline{u}_B)$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A} : \frac{\partial u_A}{\partial x}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - x^B) = 0$$

which implies:

$$\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - x^{B*})$$

For y^A:

$$\frac{\partial \mathcal{L}}{\partial y^A} : \frac{\partial u_A}{\partial y}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - x^B) = 0$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - x^{B*})$$

If $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ *is an* **interior** *feasible allocation. Then* $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ *is Pareto efficient if and only if*

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- A gives up 1 unit of y to person B in exchange for unit of x
- B is indifferent since his $MRS^B_{x,y} = 1$.
- A receives a unit of x and only needs to give one unit of y (he was willing to give two)
- We have reallocated goods to make A strictly better off without hurting B

General case

$$\max_{((x_1^1,\ldots,x_L^1),\ldots,(x_1^I,\ldots,x_L^I))} u_1(x_1^1,\ldots,x_L^1) \text{ such that } \begin{aligned} &u_2(x_1^2,\ldots,x_L^2) \geq \underline{u}_2, \\ &\vdots \\ &u_I(x_1^I,\ldots,x_L^I) \geq \underline{u}_I, \\ &x_1^1 + \cdots + x_1^I \leq \omega_1, \\ &\vdots \\ &x_L^1 + \cdots + x_L^I \leq \omega_L. \end{aligned}$$

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1, \dots, \hat{x}_L^1), \dots, (\hat{x}_1^I, \dots, \hat{x}_L^I))$ *is a feasible interior allocation. Then* $((\hat{x}_1^1, \dots, \hat{x}_L^1), \dots, (\hat{x}_1^I, \dots, \hat{x}_L^I))$ *is Pareto efficient if and only if* $((\hat{x}_1^1, \dots, \hat{x}_L^1), \dots, (\hat{x}_1^I, \dots, \hat{x}_L^I))$ *exhausts all resources* **and** *for all pairs of goods* ℓ, ℓ'*,*

$$MRS^1_{\ell,\ell'}(\hat{x}_1^1, \dots, \hat{x}_L^1) = \cdots = MRS^I_{\ell,\ell'}(\hat{x}_1^I, \dots, \hat{x}_L^I).$$

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

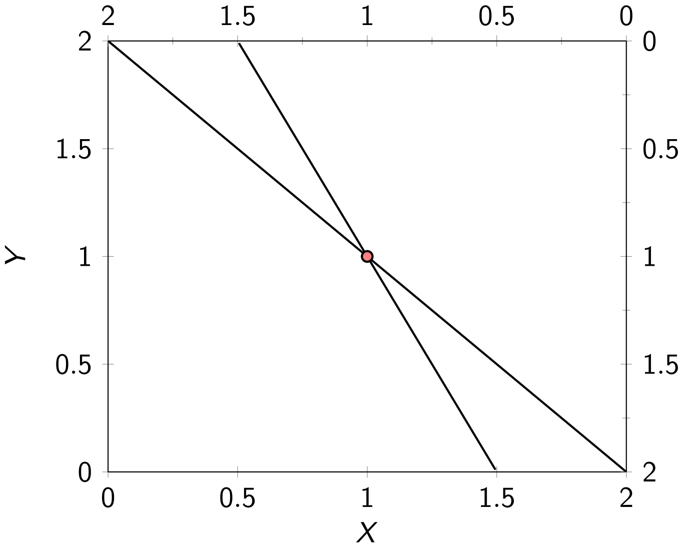
Lecture 2: General Equilibrium

Perfect substitutes

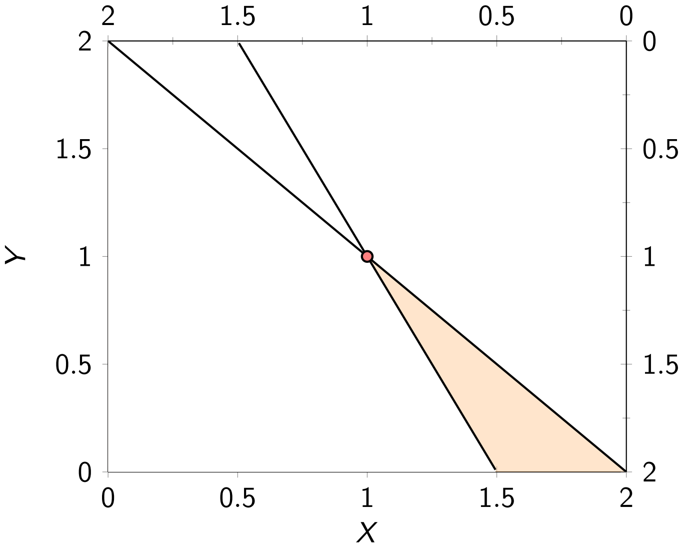
Suppose that

$$\begin{aligned} u_A(x^A, y^A) &= 2x^A + y^A \\ u_B(x^B, y^B) &= x^B + y^B \\ \omega^A &= (1, 1) \\ \omega^B &= (1, 1) \end{aligned}$$

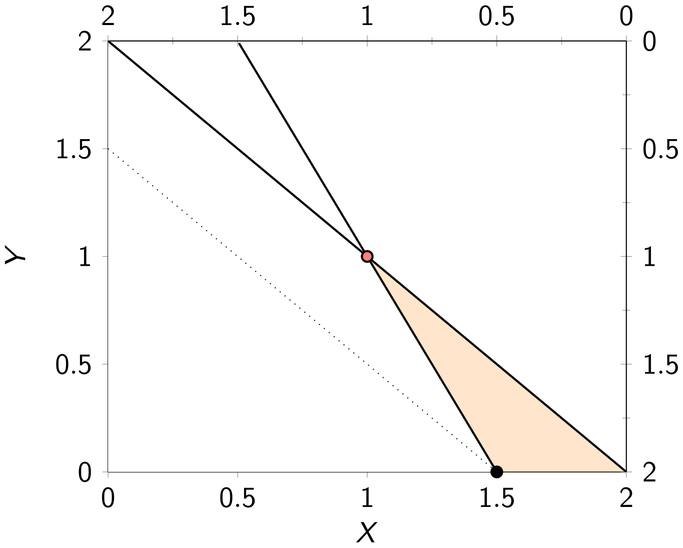
Perfect substitutes



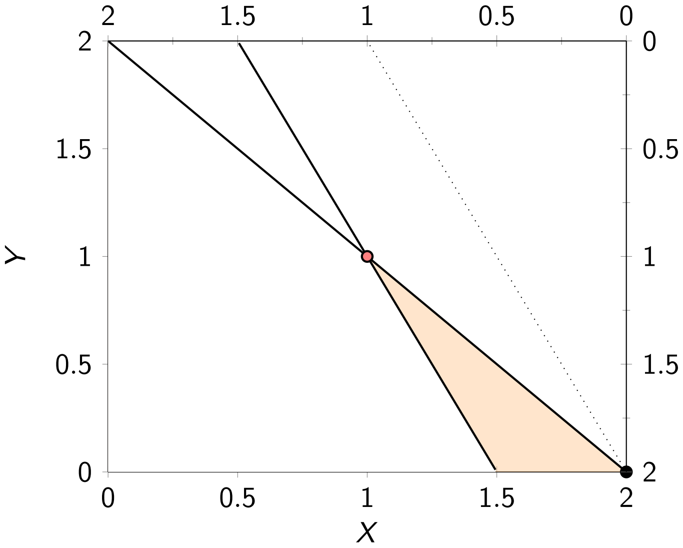
Perfect substitutes



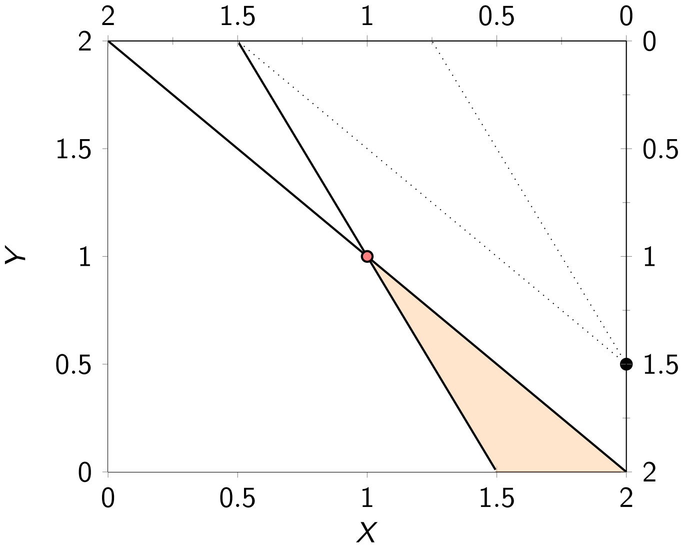
Perfect substitutes



Perfect substitutes



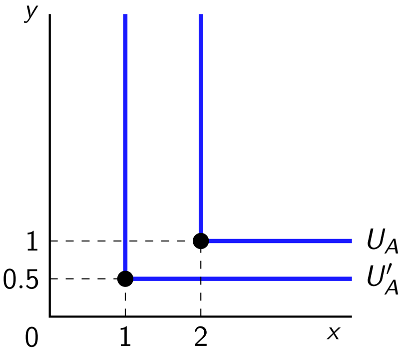
Perfect substitutes

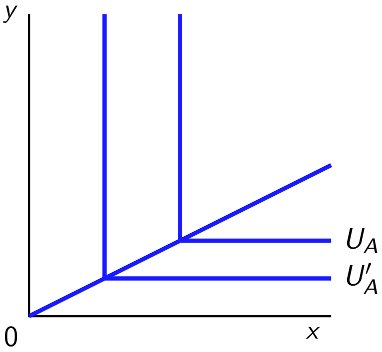


Perfect complements

Suppose that

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, 2y^A) \\
u_B(x^B, y^B) &= \min(2x^B, y^B) \\
\omega^A &= (3, 1) \\
\omega^B &= (1, 3)
\end{aligned}
$$


y
1
0.5
0
1
2
x
U_A
U'_A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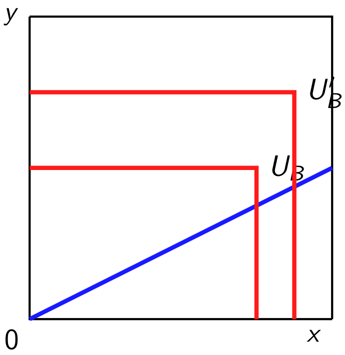
U_A
U'_A
0
x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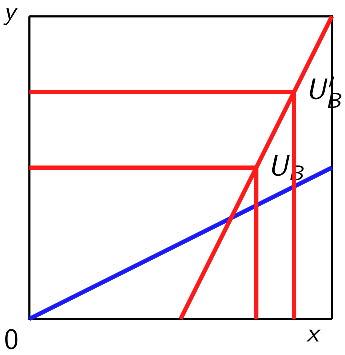
U'_B
U_B
0
x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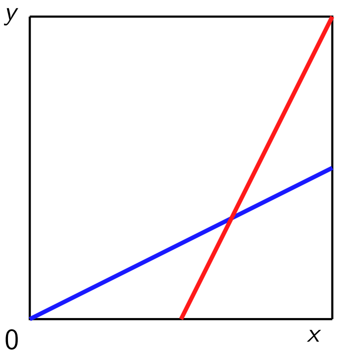
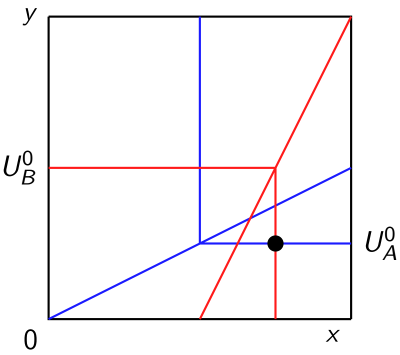
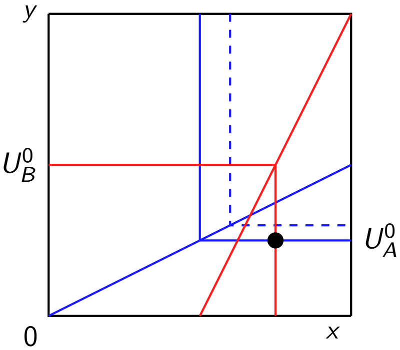
U'_B
U_B
0
x


y
0
x

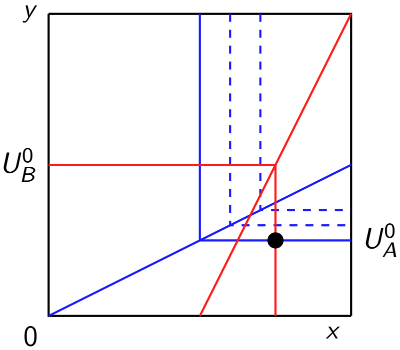


y
U_B^0
U_A^0
0
x

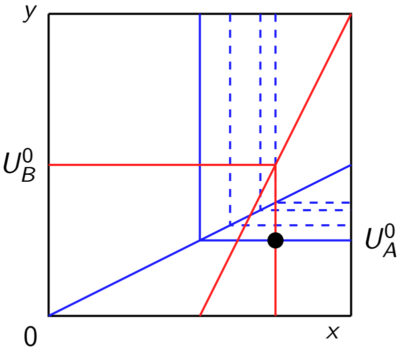
Make A as well as we can without making B worse off



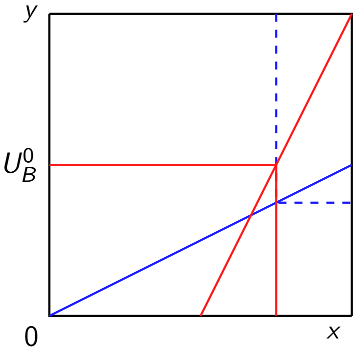
Make A as well as we can without making B worse off



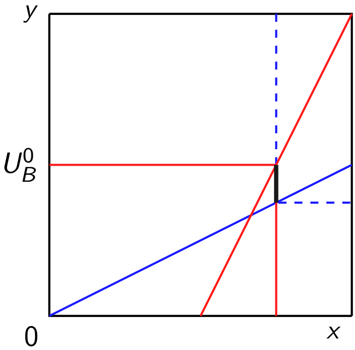
Make A as well as we can without making B worse off



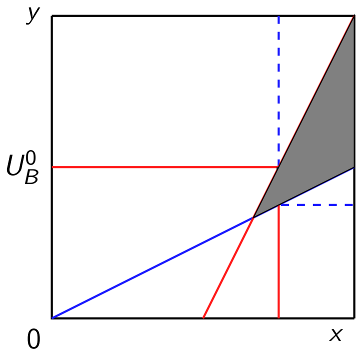
Make A as well as we can without making B worse off

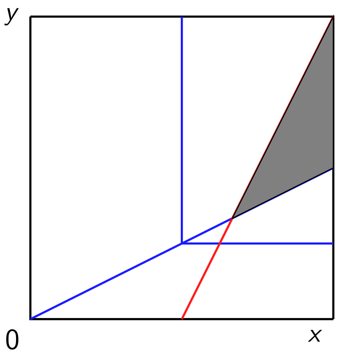


Make A as well as we can without making B worse off



Make A as well as we can without making B worse off

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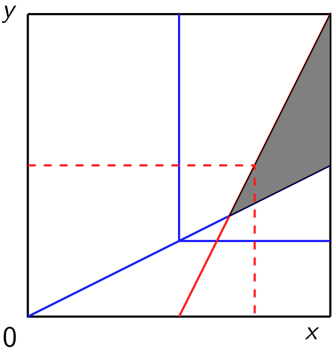
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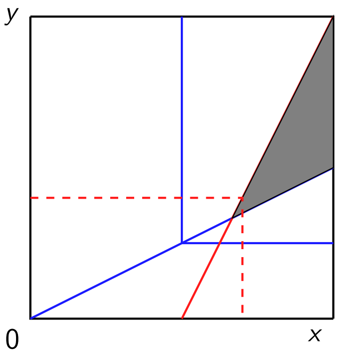
x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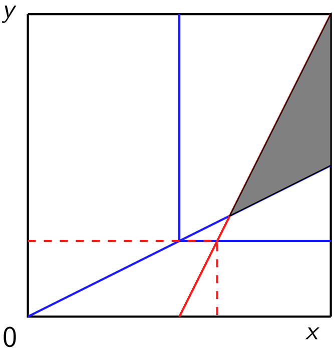
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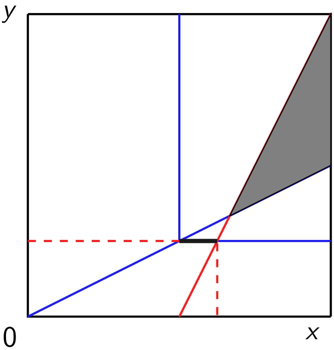
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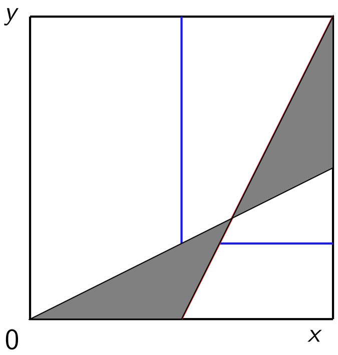
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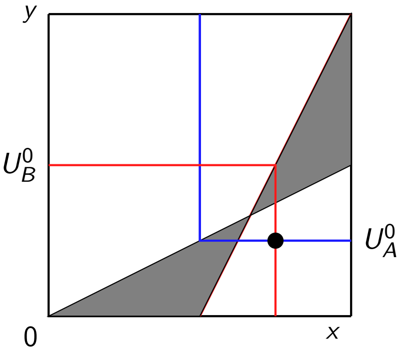
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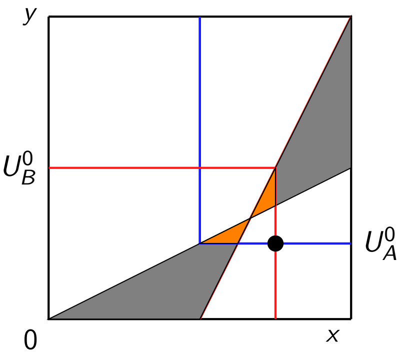
U_A^0
0
x


y
U_B^0
U_A^0
0
x

- What about: $u_A(x, y) = x^2 + y^2, u_B(x, y) = x + y$?

- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange box
- Can we say more? Not without prices