Lecture 20

Tuesday, May 04, 2021 3:15 PM

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Lecture 20: Infinitely Repeated Games

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Cournot n-firms

Bertrand n-firms

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  - Industry inverse demand curve: $p = a - b(\overbrace{q_1 + q_2 + \cdots + q_N}^{Q})$
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  - Per-firm profits: $\pi^* = \frac{(a-c)^2}{(N+1)^2 b}$ ✓
  - Note: as N grows large, $p^* \to c$ and $\pi^* \to 0$, as in PC

$N \to 1 \to$ MONOPOLIO

$$\frac{\partial \pi_i}{\partial q_i} = a - b(q_1 + \cdots + q_n) - bq_i - C = 0$$

$$\frac{a-C}{2b} - \frac{\sum_{j \neq i} q_j}{2} = q_i$$

EN UN EN SIMETRICO $\Rightarrow q_i = q^e \ \forall i$

- If firms cooperate: $\max_q = N\underset{a}{q}(a - \underset{P}{b(Nq)} - c) \to \boxed{q^c = \frac{(a-c)}{2bN}}$ $\boxed{Q = \frac{a-c}{2b}}$ $\Rightarrow q^c = \frac{1}{N} Q^{\text{MONOPOLISTA}}$
- $p^c = \frac{a+c}{2}$, higher than p^*.
- $\boxed{\pi^c} = \boxed{\frac{(a-c)^2}{4bN}}$ higher than π^*. $\pi^c = \frac{1}{N}\pi^{\text{MONOPOLISTA}}$
- But why can't each firm do this? Because NE condition is not satisfied: $\max_{q_i} \pi_i = \max_{q_i} q_i \left(a - b\left((N-1)\frac{(a-c)}{2bN} + q_i\right) - c\right) \to \boxed{q^d = \frac{(a-c)(n+1)}{4Nb}}$
- So the profits from deviating are: $\boxed{\pi^d = \frac{(n+1)^2(a-c)^2}{16bn^2}}$
- What if we repeat the game?

2-period Cournot game

- Second period: unique NE in these subgames (play the NE)
- First period: Given that NE in $t = 2 \longrightarrow$ unique SPNE is to play the NE of the stage game in both periods.
- What about 3 periods?
- What about N periods?

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  Firm i cooperates as long as it observes all other firms cooperating. If another firm cheats, firm i produces the Cournot-Nash quantity every period hereafter: **Nash reversion** (or "grim strategy")

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  - Consider the following strategy:
    1. In period t, firm i plays $q_{it} = q^c$ if $q_{-i,t-1} = \boxed{q^c}$.
    2. [redacted] Play q^* if $q_{-i,t-1} \boxed{\neq} q^c$

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Consider firm i (symmetric for all other firms)
There are two relevant subgames for firm i

- After a period in which cheating (either by himself or the other firm) has occurred
- Proposed strategy prescribes playing q^* forever (by all firms)
- This is NE of the subgame: playing q^* is a best-response to other firms playing q^*
- This satisfies SPE conditions.

- After a period when no cheating has occurred
- Proposed strategy prescribes cooperating and playing q^c, with discounted PV of payoffs $= \pi^c/(1-\delta)$
- The best other possible strategy is to play $BR_1(q^c_{-i}) \equiv q^d_i$ this period, but then be faced with $q_2 = q^*$ forever
- This yields discounted PV $= \pi^d + \delta(\pi^*/(1-\delta))$
- In order for q_c to be NE of this subgame, require $\pi^c/(1-\delta) > \pi^d + \delta(\pi^*/(1-\delta))$

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- $\delta > \frac{(n+1)^2}{n^2+6b+1}$
- This value increases with n (i.e., collusion is harder to maintain as the number of

$\boxed{\text{CASO I}}$

$h_t \neq (q_i^c, \ldots, q_i^c) \ \forall i$

$$V_{ND} = \sum_{t=0}^{\infty} \Pi(q_{ti}^*, q_{-i}^*)\,\delta^t = \Pi(q_{ti}^*, q_{-i}^*) + \sum_{t=1}^{\infty} \Pi(q_i^*, q_{-i}^*)\,\delta^t$$

$$V_D = \Pi(q_i, q_{-i}^*) + \sum_{t=1}^{\infty} \Pi(q_{ti}^*, q_i^*)\,\delta^t$$

$$V_{ND} \geq V_D$$

$$\Pi(q_{ti}^*, q_{ti}^*) + \sum_{t=1}^{\infty} \Pi(q_{ti}^*, q_{-i}^*)\,\delta^t \geq \Pi(q, q_{-i}^*) + \sum_{t=1}^{\infty} \Pi(q_{ti}^*, q_{-i}^*)\,\delta^t$$

$$\Pi(q_i^*, q_{ti}^*) \geq \Pi(q, q_{-i}^*)$$

- $\frac{(a-c)^2}{4bN(1-\delta)} > \frac{(n+1)^2(a-c)^2}{16bn^2} + \delta\left(\frac{(a-c)^2}{(N+1)^2b(1-\delta)}\right)$
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por def de E.N.

Caso 2

$$h_t = (q_i^c, \dots, q_i^c) \ \forall i$$

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$$V_D = \underbrace{\Pi(q_i, q_{-i}^c)}_{q_i \neq q_i^c} + \sum_{t=1}^{\infty} \Pi(q_i^*, q_{-i}^*)\,\delta^t = \Pi(q_i, q_{-i}^c) + \Pi(q_i^*, q_{-i}^*)\sum_{t=1}^{\infty}\delta^t$$

$$= \boxed{\Pi(q_i, q_{-i}^c)} + \Pi(q_i^*, q_{-i}^*)\frac{\delta}{1-\delta}$$

$\Rightarrow$ Futuro E.N.

$\hookrightarrow q_i = q^d$

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  - Per-firm profits: $\pi^* = 0$

- If firms cooperate: $\max_p = p\frac{a-p}{b} \to p = \frac{a+c}{2}$
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- But why can't each firm do this? Because NE condition is not satisfied. If everyone else plays $p = \frac{a+c}{2}$, I charge ε less, and essentially get the monopoly earnings all for my self
- So the profits from deviating are: $\pi^d = \frac{(a-c)^2}{4b}$
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$$V_{ND} \geq V_D$$

$$\frac{\cancel{(a-c)^2}}{4\cancel{b}N} \cdot \frac{1}{1-\delta} \geq \frac{(N+1)^2 \cancel{(a-c)^2}}{16\cancel{b}N^2} + \frac{\delta}{1-\delta}\frac{\cancel{(a-c)^2}}{(N+1)^2\cancel{b}}$$

$$\frac{1}{4N} \cdot \frac{1}{1-\delta} \geq \frac{(N+1)^2}{16N^2} + \frac{\delta}{1-\delta}\frac{1}{(N+1)^2}$$

$$\boxed{\delta^* \geq \frac{(n+1)^2}{n^2+6b+1}}$$

n CRECE $\rightarrow$ δ^* TAMBIEN CRECE

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