Lecture 20

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Lecture 20: Infinitely Repeated Games

Mauricio Romero

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Cournot n-firms

Bertrand n-firms

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  - Note: as N grows large, $p^* \to c$ and $\pi^* \to 0$, as in PC

- If firms cooperate: $\max_q = Nq(a - b(Nq) - c) \to q^c = \frac{(a-c)}{2bN}$
- $p^c = \frac{a+c}{2}$, higher than p^*.
- $\pi^c = \frac{(a-c)^2}{4bN}$, higher than π^*.
- But why can't each firm do this? Because NE condition is not satisfied: $\max_{q_i} \pi_i = \max_{q_i} q_i \left(a - b\left((N-1)\frac{(a-c)}{2bN} + q_i\right) - c\right) \to q^d = \frac{(a-c)(n+1)}{4Nb}$
- So the profits from deviating are: $\pi^d = \frac{(n+1)^2(a-c)^2}{16bn^2}$
- What if we repeat the game?

2-period Cournot game

- Second period: unique NE in these subgames (play the NE)
- First period: Given that NE in $t = 2 \longrightarrow$ unique SPNE is to play the NE of the stage game in both periods.
- What about 3 periods?
- What about N periods?

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    Firm i cooperates as long as it observes all other firms cooperating. If another firm cheats, firm i produces the Cournot-Nash quantity every period hereafter: **Nash reversion** (or "grim strategy")

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  - Consider the following strategy:
    1. In period t, firm i plays $q_{it} = q^c$ if $q_{-i,t-1} = q^c$.

       item Play q^* if $q_{-i,t-1} \neq q^c$

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Consider firm i (symmetric for all other firms)
There are two relevant subgames for firm i

- After a period in which cheating (either by himself or the other firm) has occurred
- Proposed strategy prescribes playing q^* forever (by all firms)
- This is NE of the subgame: playing q^* is a best-response to other firms playing q^*
- This satisfies SPE conditions.

- After a period when no cheating has occurred
- Proposed strategy prescribes cooperating and playing q^c, with discounted PV of payoffs $= \pi^c/(1-\delta)$
- The best other possible strategy is to play $BR_1(q^c_{-i}) \equiv q_i^d$ this period, but then be faced with $q_2 = q^*$ forever
- This yields discounted PV $= \pi^d + \delta(\pi^*/(1-\delta))$
- In order for q_c to be NE of this subgame, require $\pi^c/(1-\delta) > \pi^d + \delta(\pi^*/(1-\delta))$

- $\pi^c/(1-\delta) > \pi^d + \delta(\pi^*/(1-\delta))$
- $\frac{(a-c)^2}{4bN(1-\delta)} > \frac{(n+1)^2(a-c)^2}{16bn^2} + \delta\left(\frac{(a-c)^2}{(N+1)^2 b(1-\delta)}\right)$
- $\delta > \frac{(n+1)^2}{n^2+6b+1}$
- This value increases with n (i.e., collusion is harder to maintain as the number of firms grows)

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  - Symmetric NE prices: $p^* = c$
  - Market price: $p^* = c$
  - Per-firm profits: $\pi^* = 0$

$$P = a - bQ$$

$$Q = \frac{a-P}{b}$$

- If firms cooperate: $\max_p = p\frac{a-p}{b} \rightarrow p^m = \frac{a+c}{2}$
- $\boxed{q^c = \frac{a-c}{2bN}}$
- $\boxed{\pi^c = \frac{(a-c)^2}{4bN}}$, higher than $\pi^* = 0$ $\boxed{\pi^c = \frac{1}{N}\pi^m}$
- But why can't each firm do this? Because NE condition is not satisfied. If everyone else plays $p = \frac{a+c}{2}$, I charge ε less, and essentially get the monopoly earnings all for my self
- So the profits from deviating are: $\pi^d = \frac{(a-c)^2}{4b} = \pi^m$
- What if we repeat the game?

$$\pi^m = QP - CQ = Q(P-C)$$

$$= \frac{a-P}{b}(P-C)$$

$$\frac{\partial \pi^m}{\partial P} = -1(P-C) + \left(\frac{a-P}{b}\right)(1)$$

$$= -P + C + \frac{a}{b} - \frac{P}{b} = 0$$

$$-Pb + cb + a - P = 0$$

$$-P(b+1) + cb + a = 0$$

$$P = \frac{cb+a}{b+1}$$

$$\boxed{b=1} \quad \boxed{p^m = \frac{c+a}{2}}$$

BERTRAND

2-period [illegible] game

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- First period: Given that NE in $t = 2 \longrightarrow$ unique SPNE is to play the NE of the stage game in both periods.
- What about 3 periods?
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- Infinitely repeated game

$$(G, \infty)$$

$$S_t = \begin{cases} p^m & \text{si } h_t = (p^m, \ldots, p^m) \\ c & \text{si } \boxed{h_t \neq (p^m, \ldots, p^m)} \end{cases}$$

- Infinitely repeated game
- Are there SPNE of this game in which both firms play q^c (cooperate) each period?

CASO #1 $\rightarrow h_t \neq (p^m, \ldots, p^m)$

$$V_{ND} = \sum_{t=0}^{\infty} \pi(P_i = c, P_{-i} = c)\,\delta^t = \underbrace{\pi(P_i = c, P_{-i} = c)}_{\rightarrow 0} \sum_{t=0}^{\infty} \delta^t = 0$$

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$$V_D = \pi(P_i, P_{-i} = c) + \sum_{t=1}^{\infty} \pi(P_i = c, P_{-i} = c)\,\delta^t$$

$$= \pi(P_i, P_{-i} = c) + \cancel{\pi(P_i = c, P_{-i} = c)}^{\rightarrow 0} \sum_{t=1}^{\infty} \delta^t$$

$$= \pi(P_i, P_{-i} = c) = 0 \ \left[\text{A LO MAS}\right]$$

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$t-1$

$= \pi(P_i, \underline{P_{-i} = c}) = \underline{0}$ [A LO MAS]

$V_{ND} \geq V_D$ ✓

$\boxed{\text{CASO 2} \to h_t = (P^M, P^M, \ldots, P^M)}$

$$V_{ND} = \sum_{t=0}^{\infty} \pi(P_i = P^M, P_{-i} = P^M)\,\delta^t = \underline{\pi(P_i = P^M, P_{-i} = P^M)} \sum_{t=0}^{\infty} \delta^t -$$

$$= \underbrace{\frac{(a-c)^2}{4b}}_{\pi^m} \cdot \frac{1}{N} \cdot \frac{1}{1-\delta}$$

$$V_D = \underbrace{\pi(P_i^d, P_{-i} = P^M)} + \sum_{t=1}^{\infty} \pi(P_i = c, P_{-i} = c)\,\delta^t$$

$\to h_{t+1} = (P^M, \ldots, P^M)$

$$= \pi(P_i^d, P_{-i} = P^M) + \cancel{\pi(P_i = c, P_{-i} = c)}^{\,0} \sum_{t=1}^{\infty} \delta^t$$

firms grows)

$$\Pi(p_i^c, p_{-i}=p^c) + \Pi(p_i^c, p_{-i}=p^c)\sum_{t=1}$$

$$= \Pi(p_i^d, p_{-i}=p^m) + 0 \cdot \frac{\delta}{1-\delta}$$

$$= \Pi(p_i^d, p_{-i}=p^m)$$

$$\approx \Pi^m = \frac{(a-c)^2}{4b}$$

$$V_{ND} \geq V_D$$

$$\frac{\cancel{(a-c)^2}}{\cancel{4b}} \cdot \frac{1}{N} \cdot \boxed{\frac{1}{1-\delta}} \geq \frac{\cancel{(a-c)^2}}{\cancel{4b}} + 0\,\frac{\delta}{1-\delta}$$

$$\frac{1}{N}\frac{1}{1-\delta} \geq 1$$

$$\frac{1}{N} \geq 1-\delta$$

$$\boxed{\delta \geq 1-\frac{1}{N}}$$

$$\boxed{\delta \geq \frac{N-1}{N}}$$

$N=2 \rightarrow \delta \geq \frac{1}{2}$

$N=3 \rightarrow \delta \geq \frac{2}{3}$

$N=4 \rightarrow \delta \geq \frac{3}{4}$

A MEDIDA Q N CRECE $\rightarrow$ δ_{MIN} MAS GRANDE