Lecture 2

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Lecture2

Lecture 2: General Equilibrium

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Lecture 2: General Equilibrium

Cobb-Douglas
Using calculus
Perfect substitutes
Perfect complements

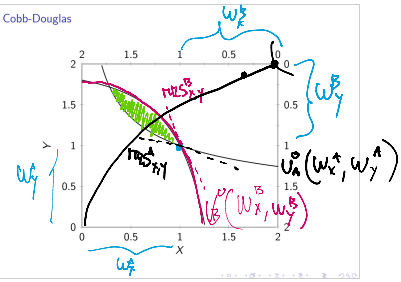
Cobb-Douglas

$$u_A(x,y) = x^{\alpha}y^{1-\alpha}$$
$$u_B(x,y) = x^{\beta}y^{1-\beta}$$

For graph suppose

$$\alpha = 0.7$$
$$\beta = 0.3$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

Cobb-Douglas



$$U^A = x^{\alpha}y^{1-\alpha}$$

$$MRS^A_{x,y} = \frac{\partial U^A/\partial x}{\partial U^A/\partial y} = \frac{\alpha x^{\alpha-1}y^{1-\alpha}}{(1-\alpha)x^{\alpha}y^{-\alpha}} = \frac{\alpha}{1-\alpha}\frac{y^{(1-\alpha-(-\alpha))}}{x^{(\alpha-(\alpha-1))}} = \frac{\alpha}{1-\alpha}\frac{y_A}{x_A}$$

Cobb-Douglas

- Indifference curves must be tangent (formalize this later)
- Thus, the MRS must be equalized across the two consumers

$$MRS^A_{x,y} = \frac{\frac{\partial x^{\alpha}y^{1-\alpha}}{\partial x}}{\frac{\partial x^{\alpha}y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha}\frac{x^{\alpha-1}y^{1-\alpha}}{x^{\alpha}y^{-\alpha}} = \frac{\alpha}{1-\alpha}\frac{y^A}{x^A}$$

$$MRS^B_{x,y} = \frac{\frac{\partial x^{\beta}y^{1-\beta}}{\partial x}}{\frac{\partial x^{\beta}y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta}\frac{x^{\beta-1}y^{1-\beta}}{x^{\beta}y^{-\beta}} = \boxed{\frac{\beta}{1-\beta}\frac{y^B}{x^B}}$$

$$\boxed{\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{y^B}{x^B}}$$

$$x^A + x^B = 2$$
$$y^A + y^B = 2$$

Cobb-Douglas

But we haven't used the fact that

$$\boxed{\begin{aligned} x^A + x^B &= \omega_x \\ y^A + y^B &= \omega_y \end{aligned}}$$

Cobb-Douglas

But we haven't used the fact that

$$\alpha y^A = \beta(\omega_y - y^A)$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\overbrace{\omega_y - y^A}^{y^B}}{\underbrace{\omega_x - x^A}_{x^B}}$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$\alpha(1-\beta)y^A(\omega_x - x^A) = (1-\alpha)\beta x^A(\omega_y - y^A)$$

$$\alpha y^A\omega_x - \alpha y^A x^A - \alpha\beta y^A\omega_x + \alpha\beta y^A x^A = \beta x^A\omega_y - \alpha\beta x^A\omega_y - y^A\beta x^A + \alpha\beta x^A y^A$$

$$\alpha y^A\omega_x - \alpha y^A x^A - \alpha\beta y^A\omega_x + y^A\beta x^A = \beta x^A\omega_y - \alpha\beta x^A\omega_y$$

$$y^A(\alpha\omega_x - \alpha x^A - \alpha\beta\omega_x + \beta x^A) = \beta x^A\omega_y - \alpha\beta x^A\omega_y$$

$$y^A = \frac{x^A(\beta\omega_y - \alpha\beta\omega_y)}{\alpha\omega_x - \alpha x^A - \alpha\beta\omega_x + \beta x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

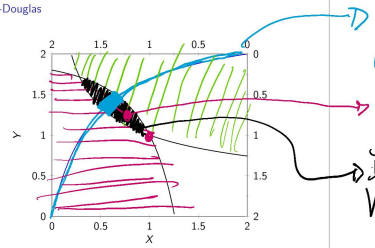
$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Then:

$$\boxed{y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}}$$

Cobb-Douglas



→ Curva Contrato
Optimos Pareto

→ Intercambios
Voluntarios

Lecture 2: General Equilibrium

Cobb-Douglas
Using calculus
Perfect substitutes
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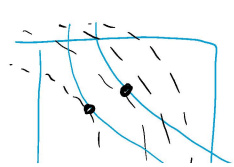
Using calculus

Essentially in this exercise we are doing the following:

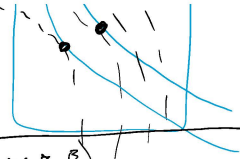
$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$



Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A,y^A) \text{ such that}$$

$$u_B(x^B,y^B) \geq \underline{u_B} = u_B(x^{B^*},y^{B^*})$$
$$x^B + x^A \leq \omega_x$$
$$y^B + y^A \leq \omega_y$$



$$\text{MAX } U^B(x^B,y^B) \quad \text{s.t} \quad U^A(x^A,y^A) = \overline{U^A}$$
$$x^A + x^B \leq \omega_x$$
$$y^A + y^B \leq \omega_y$$

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A^*},y^{A^*},x^{B^*},y^{B^*})$ is Pareto efficient if and only if it solves

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A,y^A) \text{ such that}$$
$$u_B(x^B,y^B) \geq \underline{u}_B$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

A, B, C

$$\max_{\substack{x^A,x^B,x^C \\ y^A,y^B,y^C}} U_A(x^A,y^A) \quad \text{s.t} \quad U_B(x^B,y^B) \geq \overline{U_B} = U_B^0$$
$$U_C(x^C,y^C) \geq \overline{U_C} = U_C^0$$
$$x^A + x^B + x^C \leq \omega_x$$
$$y^A + y^B + y^C \leq \omega_y$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\mathcal{L} = u_A(x^A,y^A) + \lambda u_B(\underbrace{\omega_x - x^A}_{x^B}, \underbrace{\omega_y - y^A}_{y^B}) - \underline{u}_B)$$

$$\mathcal{L} = U_A(x^A,y^A) + \lambda_1(U_B(x^B,y^B) - \overline{U_B}) + \lambda_2(x^A + x^B - \omega_x) + \lambda_3(y^A + y^B - \omega_y)$$

$$x^A + x^B = \omega_x \Rightarrow x^B = \omega_x - x^A$$
$$y^A + y^B = \omega_y \Rightarrow y^B = \omega_y - y^A$$

$$\frac{\partial \mathcal{L}}{\partial x^A} = \frac{\partial U^A}{\partial x^A} + \lambda \frac{\partial U_B}{\partial x_B}(\omega_x - x_A, \omega_y - y_A)(-1)$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A}: \frac{\partial u_A}{\partial x}(x^A,y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - x^B) = 0$$

which implies:

$$\boxed{\frac{\partial u_A}{\partial x}(x^{A^*},y^{A^*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A^*}, \omega_y - x^{B^*})}$$

$$\Rightarrow \frac{\partial U_A}{\partial x_A} \Big/ \frac{\partial U_B}{\partial x_B} = \lambda$$

For y^A:

$$\boxed{\frac{\partial \mathcal{L}}{\partial y^A}: \frac{\partial u_A}{\partial y}(x^A,y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - x^B) = 0}$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A^*},y^{A^*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A^*}, \omega_y - x^{B^*})$$

$$\Rightarrow \partial U_A / \partial y_A \Big/ \partial U_B / \partial y_B = \lambda$$

$$\frac{\frac{\partial U_A}{\partial x_A}}{\frac{\partial U_A}{\partial y_A}} = \frac{\frac{\partial U_B}{\partial x_B}}{\frac{\partial U_B}{\partial y_B}}$$

$$MRS^A_{x,y} = MRS^B_{x,y}$$

If $(x^{A^*},y^{A^*},x^{B^*},y^{B^*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A^*},y^{A^*})}{\frac{\partial u_A}{\partial y}(x^{A^*},y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A^*}, \omega_y - y^{A^*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A^*}, \omega_y - y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B^*},y^{B^*})}{\frac{\partial u_B}{\partial y}(x^{B^*},y^{B^*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A^*},y^{A^*},\omega_x - x^{A^*},\omega_y - y^{A^*})$ is an **interior** *feasible allocation. Then $(x^{A^*},y^{A^*},\omega_x - x^{A^*},\omega_y - y^{A^*})$ is Pareto efficient if and only if*

$$\frac{\frac{\partial u_A}{\partial x}(x^{A^*},y^{A^*})}{\frac{\partial u_A}{\partial y}(x^{A^*},y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A^*}, \omega_y - y^{A^*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A^*}, \omega_y - y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B^*},y^{B^*})}{\frac{\partial u_B}{\partial y}(x^{B^*},y^{B^*})}.$$

$$\boxed{MRS^A_{x,y} = MRS^B_{x,y}}$$

$$\frac{\partial U^A}{\partial x_A} \Rightarrow \frac{\text{UTILES}}{X_S}$$

$$MRS^A_{x,y} = \frac{\partial U^A/\partial x}{\partial U^A/\partial y} = \frac{\frac{\text{UTILES}}{\text{UNID } X}}{\frac{\text{UTILES}}{\text{UNID } Y}} = \frac{\text{UNID } Y}{\text{UNID } X}$$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 \geq MRS^B_{x,y} = 1$. Can we make both consumers better off?

B
DA: 1 X
DECIBE 1 Y

A
DA: 1 Y
DECIBE: 1/X

IGUAL

MESOR

Intuition

Suppose that we are at an allocation where $\boxed{MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1.}$ Can we make both consumers better off?

B
DA: 1 x
RECIBE 1 y
IGUAL

A
DA: 1 y
RECIBE: 1 x
MEJOR

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- A gives up 1 unit of y to person B in exchange for unit of x
- B is indifferent since his $MRS^B_{x,y} = 1$.
- A receives a unit of x and only needs to give one unit of y (he was willing to give two)
- We have reallocated goods to make A strictly better off without hurting B

General case

(I AGENTES, L BIENES)

$$\max_{((x_1^1,\dots,x_L^1),\dots,(x_1^I,\dots,x_L^I))} \boxed{u_1(x_1^1,\dots,x_L^1)} \text{ such that } u_2(x_1^2,\dots,x_L^2) \geq \boxed{\underline{u_2}}$$
$$\vdots$$
$$u_I(x_1^I,\dots,x_L^I) \geq \boxed{\underline{u_I}}$$
$$x_1^1 + \dots + x_1^I \leq \omega_1$$
$$\vdots$$
$$x_L^1 + \dots + x_L^I \leq \omega_L$$

I − 1

L

L + I − 1 RESTRICCIONES

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is a feasible interior allocation. Then $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is Pareto efficient if and only if $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ exhausts all resources **and** *for all pairs of goods ℓ, ℓ'*

$$\boxed{MRS^1_{\ell,\ell'}(\hat{x}_1^1,\dots,\hat{x}_L^1) = \dots = MRS^I_{\ell,\ell'}(\hat{x}_1^I,\dots,\hat{x}_L^I).}$$

$\boxed{MRS^A_{1,2} = MRS^B_{1,2}}$ ✓

$\boxed{MRS^A_{2,3} = MRS^B_{2,3}}$ ✓

$MRS^A_{1,2} = MRS^A_{2,3}$ ✗

NO ES NECESARIO

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

Lecture 2: General Equilibrium

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Using calculus
Perfect substitutes
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Perfect substitutes

Suppose that

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$\boxed{u_B(x^B, y^B) = x^B + y^B}$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

$y^A = U_A - 2x_A$

$U^B = x^B + y^B$

Suppose that

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$\boxed{u_B(x^B, y^B) = x^B + y^B}$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

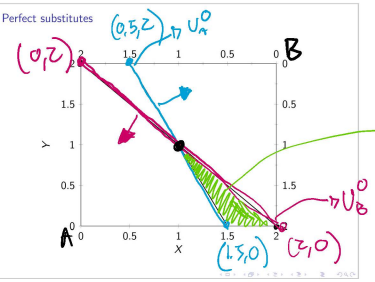
$U^B = x^B + y^B$

$\boxed{y^B = U^B - x^B}$

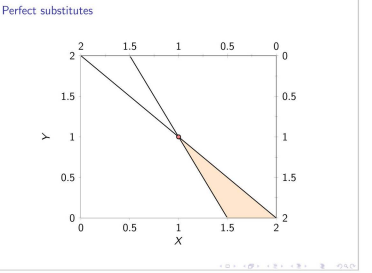
$2 - y^A = U^B - (2 - x^A)$

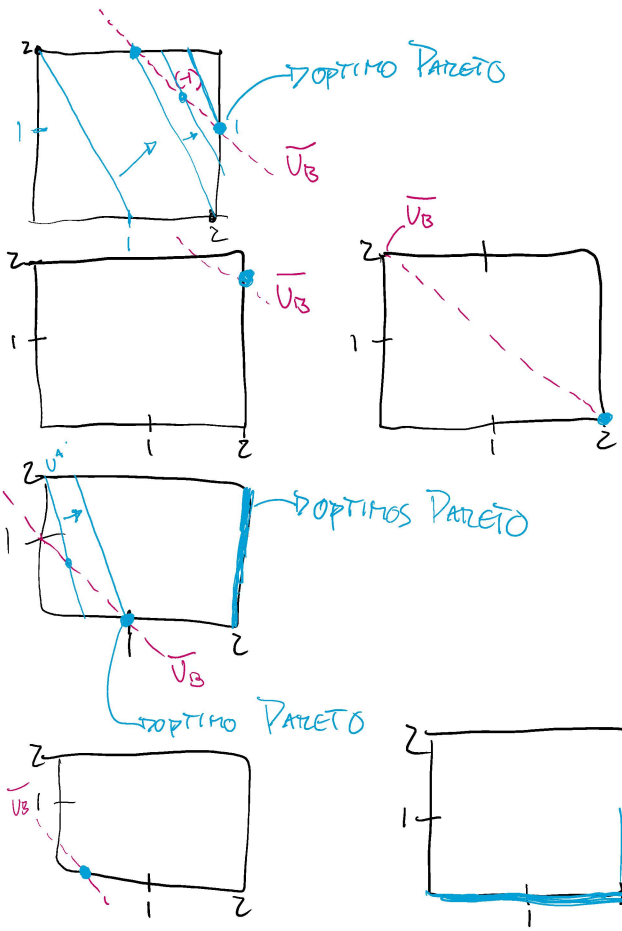
$U^B - x^A = y^A$

Perfect substitutes

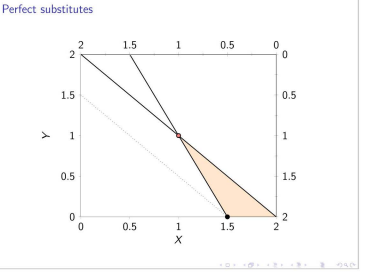


Perfect substitutes

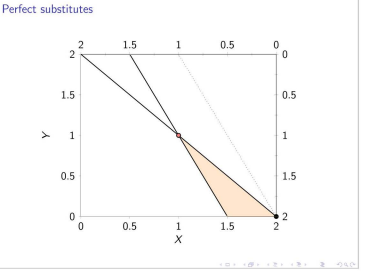




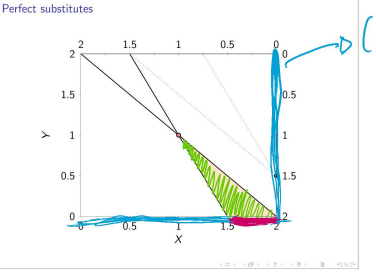
Perfect substitutes

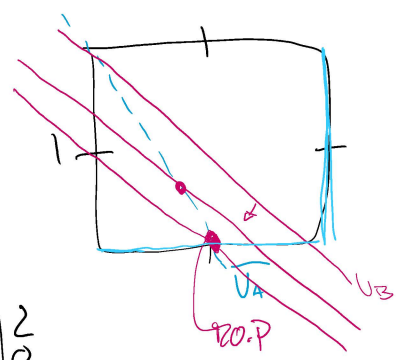


Perfect substitutes



Perfect substitutes





Lecture 2: General Equilibrium

Cobb-Douglas
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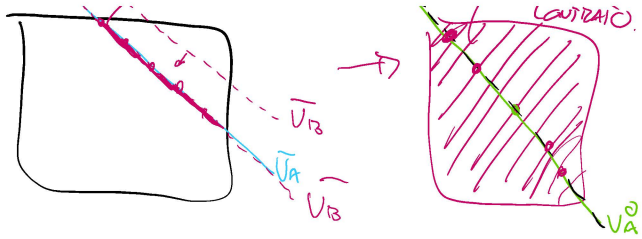
Perfect complements

$U_A = X_A + Y_A$ ≥ 0
$U_B = X_B + Y_B$

$U_A = X_A + Y_A$ ≥ 0
$U_B = 2X_B + Y_B$

$$U_A = X_A + Y_A$$
$$U_B = 2X_B + Y_B$$

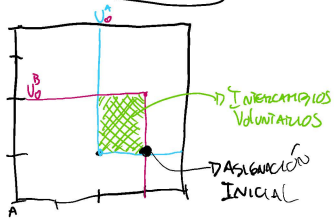


Perfect complements

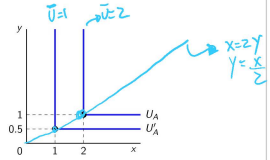
Suppose that

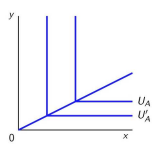
$$u_A(x^A, y^A) = \min(x^A, 2y^A)$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

ω^B → ω_B^x, ω_B^y

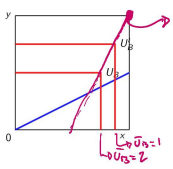


$U_A(\bar{x}, \bar{y}) = \min(\bar{x}, 2\bar{y})$





$U_B(\bar{x}, \bar{y}) = \min(2\bar{x}, \bar{y})$



$2x^B = y^B$

$\boxed{y^B = 2x^B}$

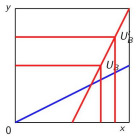
$\underbrace{4-y^A}_{y^B} = 2(\underbrace{4-x^A}_{x^B})$

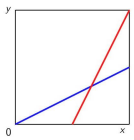
$4-y^A = 8-2x^A$

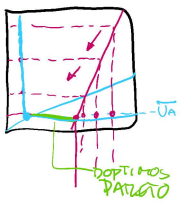
$-y^A = 4-2x^A$

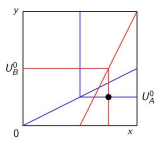
$\boxed{y^A = 2x^A - 4}$

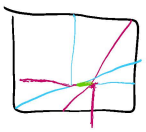
REPRESENTAN LO MISMO

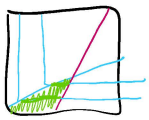




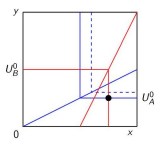


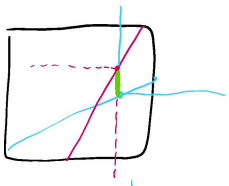


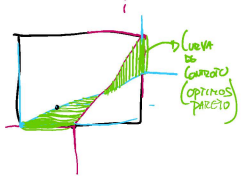




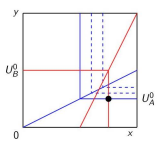
Make A as well as we can without making B worse off

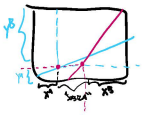




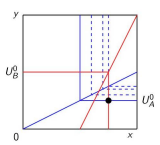


Make A as well as we can without making B worse off

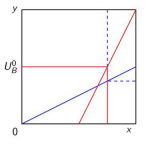




Make A as well as we can without making B worse off



Make A as well as we can without making B worse off



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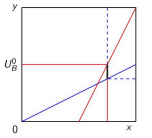

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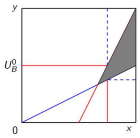
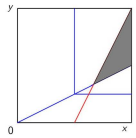
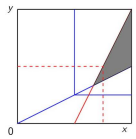
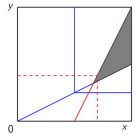
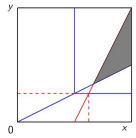
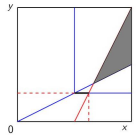


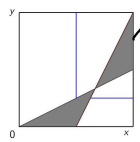




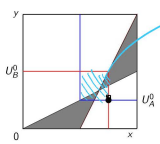




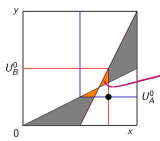




Curva de Contrato



Intercambios Voluntarios



"Acá suceden los intercambios voluntarios"

- What about $u_A(x, y) = x^2 + y^2$ $u_B(x, y) = x + y$?
- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange box
- Can we say more? Not without prices