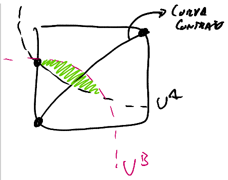
Lecture 2

Thursday, January 14, 2021 8:58 AM

Lecture2

Lecture 2: General Equilibrium

Mauricio Romero



Lecture 2: General Equilibrium

Cobb-Douglas
Using calculus
Perfect substitutes
Perfect complements

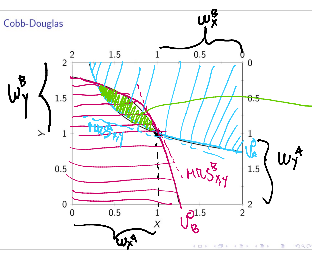
Cobb-Douglas

$$u_A(x, y) = x^{\alpha} y^{1-\alpha}$$
$$u_B(x, y) = x^{\beta} y^{1-\beta}$$

For graph suppose

$$\boxed{\alpha = 0.7}$$
$$\boxed{\beta = 0.3}$$
$$\omega^A = (1, 1) = (\omega_x^A, \omega_y^A)$$
$$\omega^B = (1, 1) = (\omega_x^B, \omega_y^B)$$

Cobb-Douglas



Intercambios Voluntarios

$$U^A = X^{\alpha} Y^{1-\alpha}$$

$$MRS^A_{XY} = \frac{\partial U^A/\partial X}{\partial U^A/\partial Y} = \frac{\alpha X^{\alpha-1} Y^{1-\alpha}}{1-\alpha\, X^{\alpha} Y^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{Y^{1-\alpha-(-\alpha)}}{X^{\alpha-(\alpha-1)}} = \frac{\alpha}{1-\alpha} \frac{Y_A}{X_A}$$

Cobb-Douglas

- Indifference curves must be tangent (formalize this later)
- Thus, the MRS must be equalized across the two consumers

$$MRS^A_{x,y} = \frac{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial x}}{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha} \frac{x^{\alpha-1} y^{1-\alpha}}{x^{\alpha} y^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{y^A}{x^A}$$

$$MRS^B_{x,y} = \frac{\frac{\partial x^{\beta} y^{1-\beta}}{\partial x}}{\frac{\partial x^{\beta} y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta} \frac{x^{\beta-1} y^{1-\beta}}{x^{\beta} y^{-\beta}} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

$$\boxed{\frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}}$$

Cobb-Douglas

But we haven't used the fact that

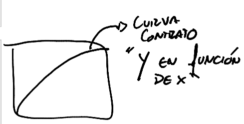
$$\boxed{x^A + x^B = \omega_x} = 2$$
$$\boxed{y^A + y^B = \omega_y} = 2$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$
$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \frac{\overbrace{\omega_y - y^A}^{Y_B}}{\underbrace{\omega_x - x^A}_{X_B}}$$



Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

$$\to (-)x^A y^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{1}{w_x - x^A}$$

$$y^A + x^A y^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{1}{w_x - x^A}$$

$$y^A\left(1 + x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{1}{w_x - x^A}\right)$$

$$y^A = x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\left(\frac{w_y - y^A}{w_x - x^A}\right)$$

$$y^A = x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta} \cdot \frac{w_y}{w_x - x^A} - x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{y^A}{w_x - x^A}$$

$$y^A + x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{y^A}{(w_x - x^A)} = x^A\left(\frac{1-\alpha}{\alpha}\right)\frac{\beta}{1-\beta}\frac{w_y}{w_x - x^A}$$

$$y^A\left(1 + x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{1}{w_x - x^A}\right) = x^A \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{w_y}{w_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

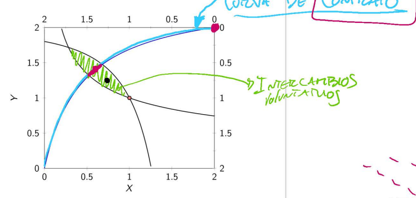
$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

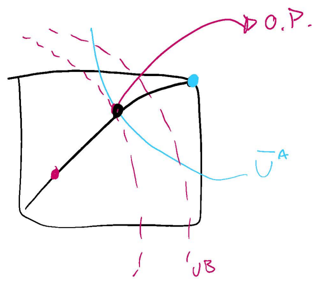
Then:

$$\boxed{y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}} \Leftarrow y^A(x^A)$$

$$y^B(x^B)$$

Cobb-Douglas





Lecture 2: General Equilibrium

Cobb-Douglas
Using calculus
Perfect substitutes
Perfect complements

Using calculus

Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$

$$u_B(x^B, y^B) \geq \underline{u}_B = u_B(x^{B*}, y^{B*})$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

$\to$ "=" Siempre cr utilidades crecientes

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient if and only if it solves

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$

$$u_B(x^B, y^B) \geq \underline{u}_B$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

$$\mathcal{L} = U_A(x^A, y^A) + \lambda_1\left(U_B(x^B, y^B) - \bar{U}_B\right) + \lambda_2\left(w_x - x^B - x^A\right) + \lambda_3\left(w_y - y^B - y^A\right)$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$w_x = x^B + x^A$$
$$w_y = y^B + y^A$$
$$\to x^B = w_x - x^A$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\mathcal{L} = u_A(x^A, y^A) + \lambda(u_B(\underbrace{\omega_x - x^A}_{X_B}, \underbrace{\omega_y - y^A}_{Y_B}) - \underline{u}_B)$$

$$x^B = \omega_x - x^A$$
$$y^B = \omega_y - y^A$$

$$\mathcal{L} = U_A(X_A, Y_A) + \lambda\left(U_B(\underbrace{\omega_x - X_A}_{X_B}, \omega_y - Y_A) - \bar{U}_B\right)$$

$$\frac{\partial \mathcal{L}}{\partial X_A} = \frac{\partial U_A}{\partial X_A} + \lambda \frac{\partial U_B}{\partial X_B}(-1) = 0 \Rightarrow \frac{\frac{\partial U_A}{\partial X_A}}{\frac{\partial U_B}{\partial X_B}} = \lambda$$

REGLA DE LA CADENA

$$\frac{\partial \mathcal{L}}{\partial Y_A} = \frac{\partial U_A}{\partial Y_A} + \lambda \frac{\partial U_B}{\partial Y_B}(-1) = 0 \;\; 2) \Rightarrow \frac{\frac{\partial U_A}{\partial Y_A}}{\frac{\partial U_B}{\partial Y_B}} = \lambda$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A} : \frac{\partial u_A}{\partial x}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - x^B) = 0$$

which implies:

$$\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - x^{B*})$$

For y^A:

$$\frac{\partial \mathcal{L}}{\partial y^A} : \frac{\partial u_A}{\partial y}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - x^B) = 0$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - x^{B*})$$

If $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

$$\frac{\frac{\partial U_A}{\partial X_A}}{\frac{\partial U_B}{\partial X_B}} = \frac{\frac{\partial U_A}{\partial Y_A}}{\frac{\partial U_B}{\partial Y_B}} \Rightarrow \frac{\frac{\partial U_A}{\partial X_A}}{\frac{\partial U_A}{\partial Y_A}} = \frac{\frac{\partial U_B}{\partial X_B}}{\frac{\partial U_B}{\partial Y_B}}$$

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is an **interior** *feasible allocation. Then $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is Pareto efficient if and only if*

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

$MRS^A_{x,y} = MRS^B_{x,y}$

BIEN Y / BIEN X

UTILES / "BIEN X"

UTILES / "BIEN Y"

$$\frac{\partial U^A}{\partial X^A} \Rightarrow \text{UNIDADES?} \Rightarrow \text{UTILES / "BIEN X"}$$

NO TIENE CONTENIDO EMPIRICO

$$\boxed{MRS^A_{X,Y} = MRS^B_{X,Y}}$$

SI TIENE CONTENIDO EMPIRICO

Intuition

Suppose that we are at an allocation where $\boxed{MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1.}$ Can we make both consumers better off?

A

DA: 1Y

RECIBE: 1X

$MRS^A_{X,Y} = 2$

MEJOR

B

DA: 1X

RECIBE: 1Y

$MRS^B_{X,Y} = 1$

IGUAL.

$MRS^A_{X,Y} = 2 \qquad MRS^B_{X,Y} = 3$

$(x^A, y^A, x^B, y^B) = O$

$\downarrow$

$(x^A - 1, y^A + 3, x^B + 1, y^B - 3) = F$

A

DA: 1X

RECIBE: 3Y

MEJOR

B

DA: 3Y

RECIBE: 1X

IGUAL

F PARETO DOMINA A O

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- *A* gives up 1 unit of *y* to person *B* in exchange for unit of *x*
- *B* is indifferent since his $MRS^B_{x,y} = 1$.
- *A* receives a unit of *x* and only needs to give one unit of *y* (he was willing to give two)
- We have reallocated goods to make *A* strictly better off without hurting *B*

General case (I PERSONAS, L BIENES

$$\max_{((x_1^1,\ldots,x_L^1),\ldots,(x_1^I,\ldots,x_L^I))} u_1(x_1^1,\ldots,x_L^I) \text{ such that } u_2(x_1^2,\ldots,x_L^2) \geq \underline{u}_2,$$
$$\vdots$$
$$u_I(x_1^I,\ldots,x_L^I) \geq \underline{u}_I,$$
$$x_1^1 + \cdots + x_1^I \leq \omega_1,$$
$$\vdots$$
$$x_L^1 + \cdots + x_L^I \leq \omega_L.$$

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ *is a feasible interior allocation. Then* $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ *is Pareto efficient if and only if* $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ *exhausts all resources* **and** *for all pairs of goods* ℓ,ℓ',

$$\boxed{MRS^1_{\ell,\ell'}(\hat{x}_1^1,\dots,\hat{x}_L^1)=\dots=MRS^I_{\ell,\ell'}(\hat{x}_1^I,\dots,\hat{x}_L^I).}$$

$\boxed{MRS^A_{1,2} = MRS^B_{1,2}}$ ✓

$\boxed{MRS^A_{2,3} = MRS^B_{2,3}}$ ✓

$MRS^A_{1,2} = MRS^A_{2,3}$

$MRS^A_{1,2} = MRS^B_{2,3}$

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

Lecture 2: General Equilibrium

Cobb-Douglas

Using calculus

Perfect substitutes

Perfect complements

Perfect substitutes

Suppose that

$$u_A(x^A,y^A) = 2x^A + y^A$$
$$u_B(x^B,y^B) = x^B + y^B$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

Curvas Ind.

$\boxed{y^A = \bar{U}_A - 2X_A}$

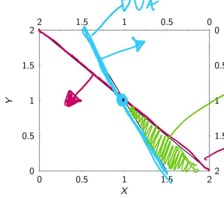
$y^B = \bar{U}_B - X_B$

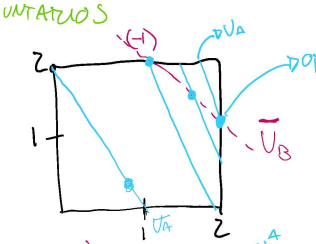
$2 - y^A = \bar{U}_B - (2 - X_A)$

$4 - \bar{U}_B - X_A = y^A$

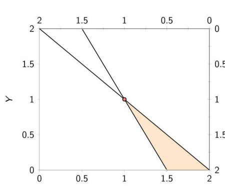
Curva Ind de B

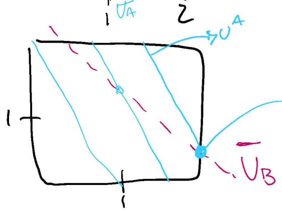
Perfect substitutes



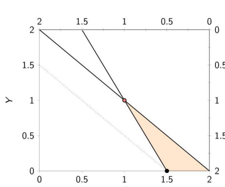


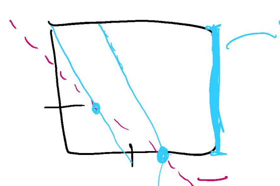
Perfect substitutes

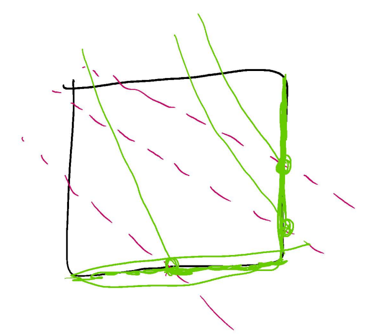




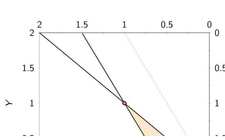
Perfect substitutes

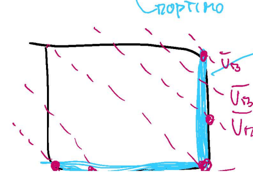


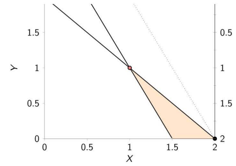


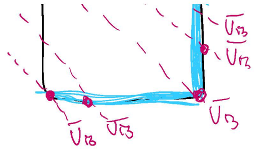


Perfect substitutes



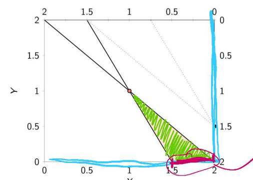






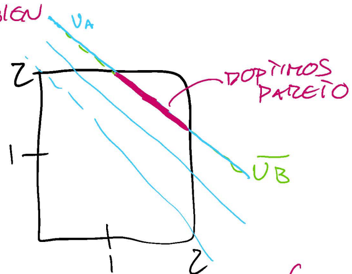
CONTRATO

Perfect substitutes



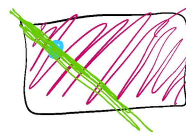
DONDE ESPERAMOS QUE INTERCAMBIEN

$$U^A = X^A + Y^A$$
$$U^B = X^B + Y^B$$



"TAREA"

$$U^A = X^A + Y^A$$
$$U^B = 2X^B + Y^B$$



CURVA CONTRATO

Lecture 2: General Equilibrium

- Cobb-Douglas
- Using calculus
- Perfect substitutes
- Perfect complements

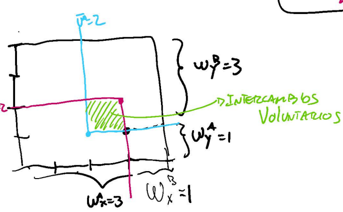
Perfect complements

Suppose that

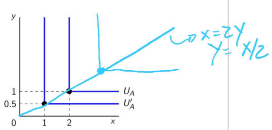
$$u_A(x^A, y^A) = \min(x^A, 2y^A)$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (3, 1)$$
$$\omega^B = (1, 3)$$

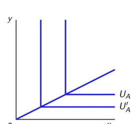
4 DE Y

4 UNIDADES X

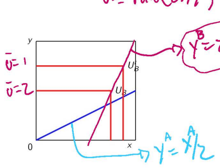


$U_A(x,y) = \min(x, 2y)$





$U_B = \min(2x^B, y^B)$

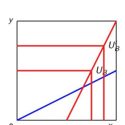


$y^B = 2x^B \Rightarrow$

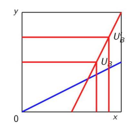
$$\underbrace{\overset{w^Y}{4} - y^A}_{y^B} = 2\underbrace{(\overset{w^X}{4} - x^A)}_{x^B}$$

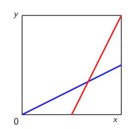
$$4 - y^A = 8 - 2x^A$$
$$-y^A = 4 - 2x^A$$
$$\boxed{y^A = 2x^A - 4}$$

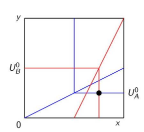
REPRESENTAN LO MISMO

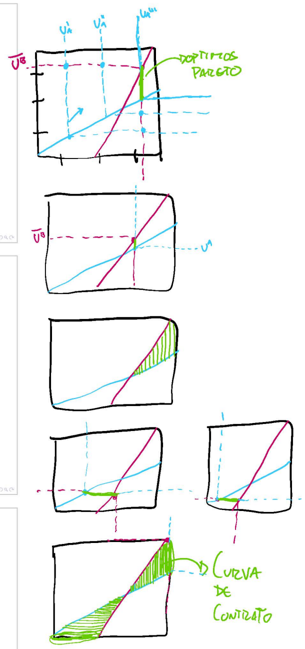


MISMO

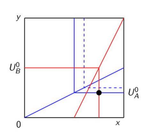




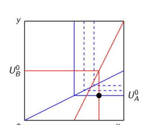




Make A as well as we can without making B worse off



Make A as well as we can without making B worse off



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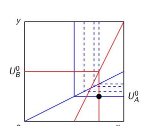
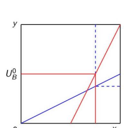


Make A as well as we can without making B worse off



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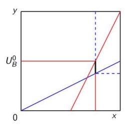
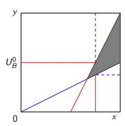
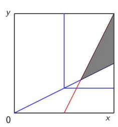
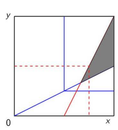
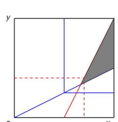


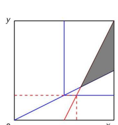
Make A as well as we can without making B worse off

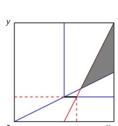


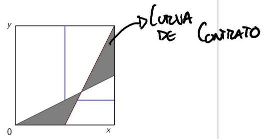


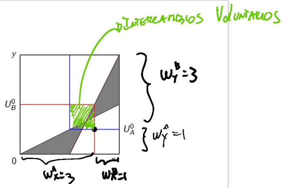


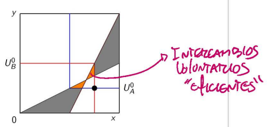












- What about $u_A(x, y) = x^2 + y^2$, $u_B(x, y) = x + y$?
- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange Triangulos
- Can we say more? Not without prices