Lecture 4

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Lecture4

Lecture 4: General Equilibrium

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Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

- The answer is going to be yes in general
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price is demand > supply), then the equilibrium is where this function stops updating

$f(x) = x$

"AJUSTE PRECIO" = "SUBASTADOR"

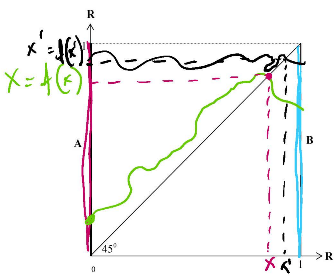
$f(p) = p'$ $\Leftarrow$ $DD = \infty$

$f(p^*) = p^*$

Lecture 4: General Equilibrium

- Is there always an equilibrium?
  - An intro to fix point theorems
  - The walrasian auctioneer
- Is the equilibrium unique?
- First welfare theorem
- Second welfare theorem

Try to draw a line from A to B without crossing the diagonal



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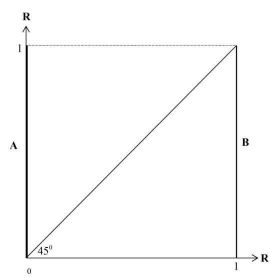
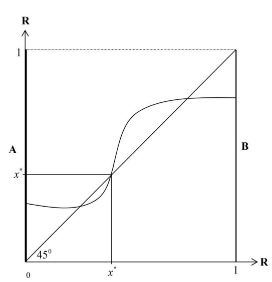


Its impossible!

For example...



There is even a theorem for this:

Theorem

For any function $f : [0,1] \to [0,1]$ that is continuous, there exists an $x^* \in [0,1]$ such that $f(x^*) = x^*$

And a more general version!

Theorem

For any function $f : \triangle^{L-1} \to \triangle^{L-1}$ that is continuous, there exists a point $p^* = (p_1^*, p_2^*, ..., p_L^*)$ such that

$$f(p^*) = p^*. \rightarrow p^* \text{ es un punto FIJO DE } f$$

where

$$\triangle^{L-1} = \{(p_1, p_2, ..., p_L) \in \mathbb{R}^L_+ \mid \sum_{l=1}^{L} p_l = 1\}$$

p^* es un PRECIO EQ

cp^* TAMBIEN " "

$\hat{p}^* = \frac{p^*}{\sum p_\ell}$ TAMBIEN $\left(c = \frac{1}{\sum p_\ell}\right)$

$$\sum_{\ell=1}^{L} \hat{p}^*_\ell = \frac{p_1}{\sum p_\ell} + \frac{p_2}{\sum p_\ell} + \frac{p_3}{\sum p_\ell} + \dots + \frac{p_L}{\sum p_\ell} = \frac{1}{\sum p_\ell}(p_1 + p_2 + \dots + p_L) = \frac{\sum p_\ell}{\sum p_\ell} = 1$$

$$\boxed{\sum_{\ell=1}^{L} \hat{p}^*_\ell = 1}$$

What was the goal again?

- Prove the existence of a general equilibrium in a market
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price if demand > supply), then the equilibrium is where this function stops updating

Lecture 4: General Equilibrium

Is there always an equilibrium?
- An intro to fix point theorems
- The walrasian auctioneer

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Excess demand

Let us define the excess demand by:

$$\underbrace{Z(p)}_{\in \mathbb{R}^L} = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} x^{*i}(p) - \sum_{i=1}^{I} w^i$$

Excess demand

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \overset{DD}{\boxed{\sum_{i=1}^{I} x^{*i}(p)}} - \overset{OO}{\boxed{\sum_{i=1}^{I} w^i}}$$

since $x^{*i}(p)$ is the demand (i.e., consumers are already maximizing) then we have the following result:

Remark

$p \in \mathbb{R}^n_{++}$ is a competitive equilibrium if and only if $Z(p) = \vec{0}$ $= \underbrace{(0, 0, \dots, 0)}_{L \text{ ENTRADAS.}}$

Excess demand

Z(p) has the following properties

1. Is continuous in p

$Z(p) = (Z_1(p), \dots, Z_L(p))$

↳ CONTINUO $\Rightarrow Z_\ell(p) = \sum_{i=1}^{I} x_\ell^i(p) - \sum_{i=1}^{I} w_\ell^i$

↳ SIEMPRE Y CUANDO $x_\ell^i(p)$ SEA CONTINUO $\forall i$

2. Is homogeneous of degree zero

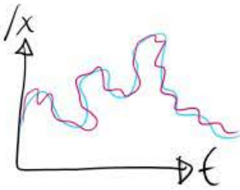
$Z(\lambda p) = Z(p)$

3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law) $\forall p$ (ASI P NO SEA DE EQ)

$p_1 Z_1(p) + p_2 Z_2(p) + \dots + p_L Z_L(p) = 0$

$\underbrace{p_1 Z_1(p)}_{>0}$ + $p_2 Z_2(p) = 0$

DEFICIT COMERCIAL



Excess demand

Z(p) has the following properties

1. Is continuous in p

2. Is homogeneous of degree zero

3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law) — Think about this!

Excess demand

We said we want to update prices in a "logical" way. If excess demand is positive, then increase prices...

$$T(p) = p + Z(p)$$
$$= (p_1 + Z_1(p), p_2 + Z_2(p), \dots, p_L + Z_L(p))$$
$$= (p_1 + \max(Z_1(p), 0), p_2 + \max(Z_2(p), 0), \dots, p_L + \max(Z_L(p), 0))$$

$$T(p) = \frac{(p_1 + \max(Z_1, 0), p_2 + \max(Z_2, 0), \dots, p_L + \max(Z_L, 0))}{\sum_{\ell=1} p_\ell + \max(Z_\ell, 0)}$$

Excess demand

We said we want to update prices in a "logical" way. If excess demand is positive, then increase prices...

$$p' = p + Z(p)$$

But what if $p' < 0$? Ok then

$$T(p) = \frac{1}{\sum_{l=1}^{L} p_l + \max(0, Z_l(p))}(p_1 + \max(0, Z_1(p)), \max(0, Z_2(p)), \dots, p_l + \max(0, Z_l(p)))$$

Excess demand

- $\boxed{T \text{ is continuous}}$
- Thus we can apply the fix point theorem
- Therefore there exists a $\boxed{p^*}$ such that $T(p^*) = p^*$ → P^* ES UN PRECIO EQ
- Then $Z(p^*) = \vec{0} = (Z_1 = 0, Z_2 = 0, \dots, Z_L = 0)$

$$T(p^*) = \frac{(p_1^* + \max(Z_1, 0), \dots, p_L^* + \max(Z_L, 0))}{\sum_{\ell=1}^{I} p_\ell + \max(Z_\ell, 0)} = (p_1^*, \dots, p_L^*)$$

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $T(p^*) = p^*$
- Then $Z(p^*) = 0$ (why?)

So when does it break down?

- We needed demand to be continuous!

Weird case - no equilibrium

$$u_A(x^A, y^A) = \min(x^A, y^A)$$
$$u_B(x^B, y^B) = \max(x^B, y^B)$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

DDD^B

$P_y = 1$

$P_x < 1$

TODO X

$X(P_x) = \frac{I^A}{P_x} = \frac{P_x \cdot 1 + P_y \cdot 1}{P_x} = 1 + \frac{1}{P_x} > Z = \infty$ AGG

$\frac{1}{P_x} > 1$

$P_x > 1$

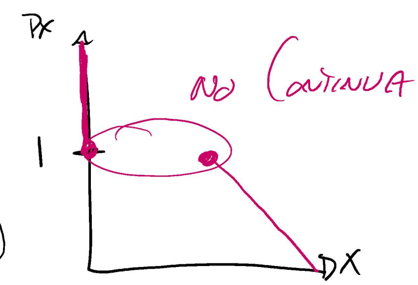
TODO Y

$X(P_x) = 0$

$P_x = 1$

TODO X ó TODO Y

$X(P_x) = 0$ ó I/P_x



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- prices are positive (why?)
- normalize $p_x = 1$

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Weird case - no equilibrium

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- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible $Y^b = \frac{1}{p_y} + 1$
- if $p_y > 1$ then B wants to demand as much of x as possible $X^b = p_y + 1$
- if $p_y = 1$ then B either demands two units of X or two units of Y, but A demands at least one unit of each good

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Is the equilibrium unique?

We have seen it is not

Lecture 4: General Equilibrium

- Is there always an equilibrium? SI, SI LAS DD SON CONTINUAS
- Is the equilibrium unique? NO NECESARIAMENTE
- First welfare theorem
- Second welfare theorem

Lecture 4: General Equilibrium

- Is there always an equilibrium?
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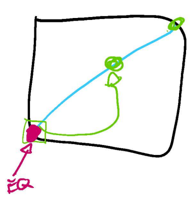
First welfare theorem

Theorem
Consider any pure exchange economy. Suppose that all consumers have weakly monotone utility functions. Then if (x^*, p) is a competitive equilibrium, then x^* is a Pareto efficient allocation.



Proof

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Assume that $(p, (x^1, x^2, ..., x^I))$ is a competitive equilibrium but that $(x^1, x^2, ..., x^I)$ is not Pareto efficient

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- is feasible
- pareto dominates $(x^1, x^2, ..., x^I)$

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- is feasible
- pareto dominates $(x^1, x^2, ..., x^I)$

In other words:

1. $\sum_{i=1}^{I} \hat{x}^i = \sum_{i=1}^{I} w^i$
2. For all i, $u^i(\hat{x}^i) \geqslant u^i(x^i)$
3. For some i^*, $u^{i^*}(\hat{x}^{i^*}) > u^{i^*}(x^{i^*})$

Proof

GASTO INGRESO

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \hat{x}^{i^*} > p \cdot w^{i^*}$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \hat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\hat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \hat{x}^i \geqslant p \cdot w^i$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $\boxed{p \cdot \hat{x}^{i^*} > p \cdot w^{i^*}}$
  - Otherwise, why didn't i^* pick $\hat{x}^{i^*}$ to begin with
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Adding over all agents we get:

$$\underline{\sum_{i=1}^{I} p \cdot \hat{x}^i} \boxed{>} \underline{\sum_{i=1}^{I} p \cdot w^i}$$

$a > b$

$c \geq d$

$a + c > b + d$

Proof

By definition of an equilibrium we have that

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Adding over all agents we get:

$$\sum_{i=1}^{I} p \cdot \hat{x}^i > \sum_{i=1}^{I} p \cdot w^i$$

Which in turn implies

$$\boxed{p \cdot \boxed{\sum_{i=1}^{I} \hat{x}^i} > p \cdot \boxed{\sum_{i=1}^{I} w^i}}$$

Contradiction

$$\boxed{\sum_{i=1}^{I} \hat{x}^i = \sum_{i=1}^{I} w^i}$$

Proof

By definition of an equilibrium we have that

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Adding over all agents we get:

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Which in turn implies

$$p \cdot \sum_{i=1}^{I} \hat{x}^i > p \cdot \sum_{i=1}^{I} w^i$$

Which contradicts what Condition 1 in the previous slide implies.

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---

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- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations

$$z(p) = (z_1(p), \ldots, z_L(p))$$

$$\sum_{i=1}^{I} x_i^\ell(p) - \sum_{i=1}^{I} w_i^\ell$$

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- How about the opposite?

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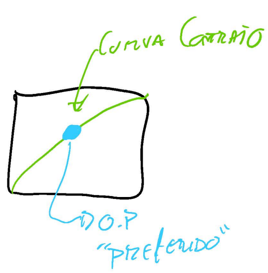
  - Not in general...



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- How about the opposite?
  - Maybe we “like” one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?

  - Not in general... but what if we allow for a redistribution of resources?

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Second welfare theorem

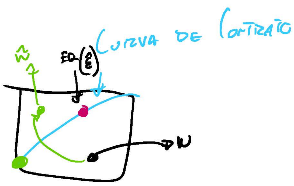
Theorem

Given an economy $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, w^i\right)_{i\in\mathcal{I}} \right\rangle$ where all consumers have weakly monotone, $\boxed{\text{quasi-concave}}$ utility functions. If $\boxed{\left(x^1, x^2, ..., x^I\right)}$ is a Pareto optimal allocation then there exists a redistribution of resources $\left(\widehat{w}^1, \widehat{w}^2, ..., \widehat{w}^I\right)$ and some prices $p = (p_1, p_2, ..., p_L)$ such that:

→ O.P.

1. $\boxed{\sum_{i=1}^{I} \widehat{w}^i = \sum_{i=1}^{I} w^i}$ → $\hat{w}$ ES UNA REDISTRIBUCIÓN DE LAS DOTACIONES INICIALES.
2. $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ *is a competitive equilibrium of the economy* $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, \widehat{w}^i\right)_{i\in\mathcal{I}} \right\rangle$

ANTES = ANTES



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- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution?

- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea

  - Ok... but *how* can we do this redistribution? Not taxes, since they produce dead-weight loss

- In contrast to the first welfare theorem, we require an additional assumption that all utility functions are quasi-concave.
- What if they are not? consider the following:

NO ES QUASI-CONCAVIDAD

$$u_A(x, y) = \boxed{\max\{x, y\}}$$
$$u_B(x, y) = \min\{x, y\}$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

In this example, all points in the Edgeworth Box are Pareto efficient. However we cannot obtain any of these points as a competitive equilibrium after transfers.