Lecture 4

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Lecture4

Lecture 4: General Equilibrium

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Lecture 4: General Equilibrium

- Is there always an equilibrium?
- Is the equilibrium unique?
- First welfare theorem
- Second welfare theorem

Lecture 4: General Equilibrium

- Is there always an equilibrium?
- Is the equilibrium unique?
- First welfare theorem
- Second welfare theorem

- The answer is going to be yes in general
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price is demand > supply), then the equilibrium is where this function stops updating

Lecture 4: General Equilibrium

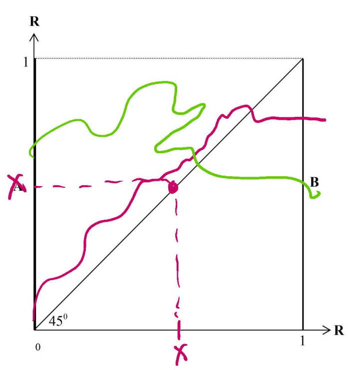
Is there always an equilibrium?
An intro to fix point theorems
The walrasian auctioneer

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Try to draw a line from A to B without crossing the diagonal



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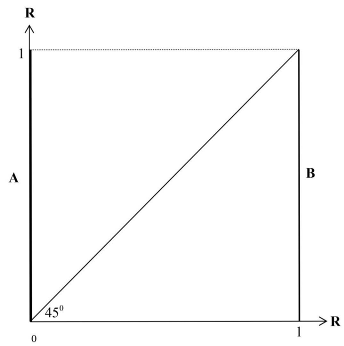
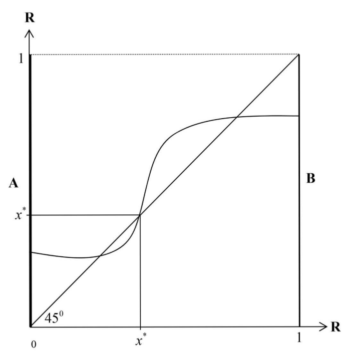


Its impossible!

For example...



There is even a theorem for this:

Theorem

For any function $f : [0,1] \to [0,1]$ *that is continuous, there exists an* $x^* \in [0,1]$ *such that* $f(x^*) = x^*$

OSEA x^* ES UN
PUNTO FIJO

And a more general version!

Theorem

For any function $f : \boxed{\triangle^{L-1} \to \triangle^{L-1}}$ *that is continuous, there exists a point* $p^* = (p_1^*, p_2^*, ..., p_L^*)$ *such that*

$$\boxed{f(p^*) = p^*.}$$

P^* es un punto fijo

where

$$\triangle^{L-1} = \{(p_1, p_2, ..., p_L) \in \mathbb{R}_+^L \mid \sum_{l=1}^{L} p_l = 1\}$$

Normalizar precios

P^* es de EQ

$p^\alpha = \frac{P^*}{\Sigma P}$ también es EQ

$$\boxed{\sum_{l=1}^{L} p_l^\alpha = 1}$$

What was the goal again?

- Prove the existence of a general equilibrium in a market
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price if demand > supply), then the equilibrium is where this function stops updating

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Is there always an equilibrium?

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First welfare theorem

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Excess demand

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} x^{*i}(p) - \sum_{i=1}^{I} w^i$$

Excess demand

$DZ = \boxed{DD} - OO$

$\to \Sigma$ DD ~~MARSHA~~ MARSHALLIANAS.

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} x^{*i}(p) - \sum_{i=1}^{I} w^i$$

since $x^{*i}(p)$ is the demand (i.e., consumers are already maximizing) then we have the following result:

Remark

$p \in \mathbb{R}^n_{++}$ is a competitive equilibrium if and only if $Z(p) = 0$ $\to \underbrace{(0, 0, \ldots, 0)}_{L \text{ ENTRADAS}}$

Excess demand

Z(p) has the following properties

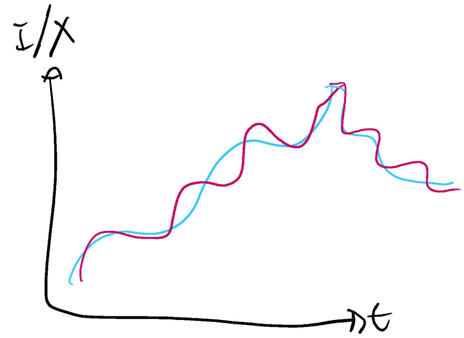
$Z(P) = (Z_1(P), Z_2(P) \ldots, Z_L(P))$

$\to \sum x_i^*(P) - \sum W_i$

1. Is continuous in p
2. Is homogeneous of degree zero
3. $\boxed{p \cdot Z(p) = 0}$ (this is equivalent to Walra's law)

$Z(\lambda P) = Z(P)$

$$\boxed{\begin{array}{l} P \cdot Z(P) = 0 \quad \forall P \\ P_1 \cdot Z_1(P) + P_2 \cdot Z_2(P) + \ldots + P_L \cdot Z_L(P) = 0 \end{array}}$$



Excess demand

Z(p) has the following properties

1. Is continuous in p
2. Is homogeneous of degree zero
3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law) — Think about this!

Excess demand

We said we want to update prices in a “logical” way. If excess demand is positive, then increase prices...

Excess demand

We said we want to update prices in a “logical” way. If excess demand is positive, then increase prices...

$$\boxed{p' = p + Z(p)} = (p_1', p_2', \ldots, p_L') = (p_1, \ldots, p_L) + (Z_1(p), \ldots, Z_L(p))$$

But what if $\boxed{p' < 0}$? Ok then

$$T(p) = \frac{1}{\sum_{i=1}^{L} p_l + max\left(0, Z_l(p)\right)} \left(p_1 + \max\left(0, Z_1(p)\right), \max\left(0, Z_2(p)\right), \ldots, p_L + \max\left(0, Z_L(p)\right)\right)$$

A $\quad Z_A > 0 \quad p_A' = p_A + \underset{\max(Z_A, 0)}{Z_A}$

B $\quad Z_B < 0 \quad p_B' = p_B + \underset{\max(Z_B, 0)}{0}$

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a $\boxed{p^*}$ such that $\boxed{T(p^*)} = \boxed{p^*}$

$\Rightarrow T(p^*) = (\mathrm{MAX}(0, \cancel{z_1(p)}^{\,0}) + p_1, \mathrm{MAX}(0, \cancel{z_2(p)}^{\,0}) + p_2, \ldots) = (p_1, \ldots, p_L) = (p_1, \ldots, p_L)$

- Then $Z(p^*) = 0$

$\Rightarrow p^*$ ES PRECIO DE EQ!

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $T(p^*) = p^*$
- Then $Z(p^*) = 0$ (why?)

So when does it break down?

- We needed demand to be continuous!

Weird case - no equilibrium

$$\begin{aligned} u_A(x^A, y^A) &= \min(x^A, y^A) \\ u_B(x^B, y^B) &= \max(x^B, y^B) \\ \omega^A &= (1,1) \\ \omega^B &= (1,1) \end{aligned}$$

Weird case - no equilibrium

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- prices are positive (why?)
- normalize $p_x = 1$

Weird case - no equilibrium

$$u_A(x^A, y^A) = \min(x^A, y^A)$$
$$\boxed{u_B(x^B, y^B) = \max(x^B, y^B)} \rightarrow \begin{matrix} 0 \text{ TODO } x \\ 0 \text{ TODO } y \end{matrix}$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

- prices are positive (why?)
- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible
  $\boxed{Y^b = \frac{1}{p_y} + 1} = \frac{P_x \cdot 1 + P_y \cdot 1}{P_y} = \frac{1}{P_y} + 1 > 2$

  ∞ AGREGADA

Weird case - no equilibrium

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- if $p_y < 1$ then B wants to demand as much of y as possible
  $Y^b = \frac{1}{p_y} + 1$
- if $p_y > 1$ then B wants to demand as much of x as possible
  $X^b = p_y + 1 = \frac{P_x + P_y}{P_x} = 1 + P_y > 2$

  ∞ AGREGADA

Weird case - no equilibrium

$$u_A(x^A, y^A) = \min(x^A, y^A)$$
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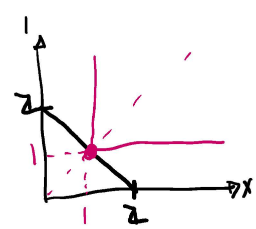
- prices are positive (why?)
- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible
  $Y^b = \frac{1}{p_y} + 1$
- if $p_y > 1$ then B wants to demand as much of x as possible
  $X^b = p_y + 1$
- if $p_y = 1$ then B either demands two units of X or two units of Y, but A demands at least one unit of each good

DD → A) $P_x = 1$, $P_y = 1$, $U_A = \text{MIN}(x, Y)$



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Is the equilibrium unique?

We have seen it is not

Lecture 4: General Equilibrium

- Is there always an equilibrium? (CASI SIEMPRE)
- Is the equilibrium unique? NO NECESARIAMENTE
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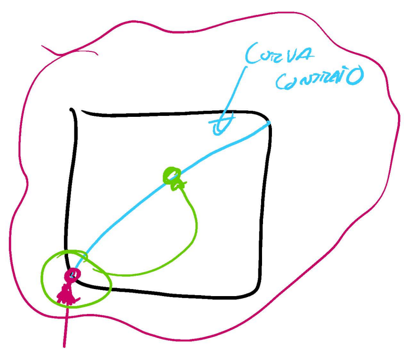
First welfare theorem

Theorem

Consider any pure exchange economy. Suppose that all consumers have weakly monotone utility functions. Then if (x^*, p) is a competitive equilibrium, then x^* is a Pareto efficient allocation.



Proof

By contradiction:

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By contradiction:
Assume that $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ is a competitive equilibrium but that $\left(x^1, x^2, ..., x^I\right)$ is $\boxed{\text{not}}$ Pareto efficient

Proof

By contradiction:
Assume that $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ is a competitive equilibrium but that $\left(x^1, x^2, ..., x^I\right)$ is not Pareto efficient
Then there is an allocation $\left(\widehat{x}^1, \widehat{x}^2, ..., \widehat{x}^I\right)$ such that

- is feasible
- pareto dominates $\left(x^1, x^2, ..., x^I\right)$

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In other words:

1. $\boxed{\sum_{i=1}^{I} \widehat{x}^i = \sum_{i=1}^{I} w^i}$
2. For all i, $u^i\left(\widehat{x}^i\right) \geqslant u^i\left(x^i\right)$
3. For some i^*, $u^{i^*}\left(\widehat{x}^{i^*}\right) > u^{i^*}\left(x^{i^*}\right)$

Proof

INGRESO

By definition of an equilibrium we have that GASTO

- Condition 3 in the previous slide implies $\boxed{p \cdot \widehat{x}^{i^*}} > \boxed{p \cdot w^{i^*}}$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \widehat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\widehat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \widehat{x}^i \geqslant p \cdot w^i$

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Adding over all agents we get:

$$\sum_{i=1}^{I} p \cdot \widehat{x}^i \boxed{>} \sum_{i=1}^{I} p \cdot w^i$$

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Which in turn implies

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Which contradicts what Condition 1 in the previous slide implies.

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it

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- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations

$\forall p$

$$\boxed{p \cdot z(p) = 0}$$

$$\boxed{z(p^*) = 0}$$

EQ

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it

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- How about the opposite?

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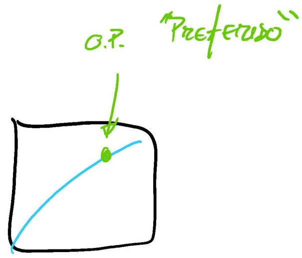
- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?
  - Not in general... but what if we allow for a redistribution of resources?



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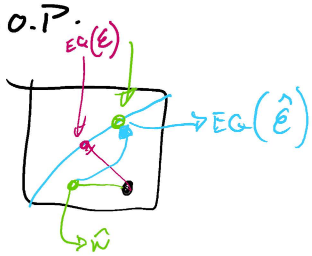
Theorem

Given an economy $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, w^i\right)_{i\in\mathcal{I}} \right\rangle$ where all consumers have weakly monotone, $\boxed{\text{quasi-concave}}$ utility functions. If $\boxed{\left(x^1, x^2, ..., x^I\right)}$ is a Pareto optimal allocation then there exists a redistribution of resources $\left(\widehat{w}^1, \widehat{w}^2, ..., \widehat{w}^I\right)$ and some prices $p = \left(p_1, p_2, ..., p_L\right)$ such that:

O.P.

1. $\boxed{\sum_{i=1}^I \widehat{w}^i = \sum_{i=1}^I w^i}$ REDISTRIBUCIÓN
2. $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ *is a competitive equilibrium of the economy* $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, \widehat{w}^i\right)_{i\in\mathcal{I}} \right\rangle$

mismo ≠



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- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution?

- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution? Not taxes, since they produce dead-weight loss

- In contrast to the first welfare theorem, we require an additional assumption that all utility functions are quasi-concave.
- What if they are not? consider the following:

NO ES QUASI CONCAVO

$$u_A(x, y) = \max\{x, y\}$$
$$u_B(x, y) = \min\{x, y\}$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

In this example, all points in the Edgeworth Box are Pareto efficient. However we cannot obtain any of these points as a competitive equilibrium after transfers.