Lecture 6

Tuesday, January 26, 2021 5:58 PM

Lecture6

Lecture 6: General Equilibrium

Mauricio Romero

A few things I forgot to say about economies with production

Robinson Crusoe

Two firms

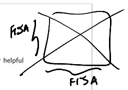
General Economies with Many Consumers and Production

A few things I forgot to say about economies with production

Robinson Crusoe

Two firms

General Economies with Many Consumers and Production



- With production, Edgeworth box illustrations are no longer helpful
- Depending on the production plan, the size of the box can change
- Instead we work with what is called a production possibilities frontier

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General Economies with Many Consumers and Production

- Imagine the problem of Robinson Crusoe, living alone in an island. He is the only producer and the only consumer

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Suppose that the consumer (Robinson) has a utility function:

$$\boxed{u(L, x),}$$

where x are coconuts. There is one firm (Robinson) that can convert labor to coconuts:

$$\boxed{f(L_x)}$$

The endowment is $(0, \bar{L})$.

What is the Pareto optimal allocation in this economy?

What is the Pareto optimal allocation in this economy?

$$\max u(L, x) \text{ such that } x \le f(L_x)$$
$$L_x + L \le \bar{L}.$$

$\Rightarrow \boxed{x = f(L_x)} \Rightarrow x = f(\underbrace{\bar{L} - L}_{L_x})$

$L_x + L = \bar{L}$

$L = \bar{L} - L_x$

$\Rightarrow \max_{L_x} U(\bar{L} - L_x, f(L_x))$

$L_x + L = \bar{L}$

$L = \bar{L} - L_x$

$\Rightarrow \max_{L_x} U(\bar{L} - L_x, f(L_x))$

$\Rightarrow \max_{L} U(L, f(\bar{L} - L))$

Or equivalently

$$\max u(L, f(\bar{L} - L))$$

Or equivalently

OCIO

TRABAJO

$$\max u(L, f(\bar{L} - L))$$

We can solve this either using calculus or graphically

$$\frac{\partial}{\partial L} = \frac{\partial U}{\partial L} + \frac{\partial U}{\partial X} \cdot \frac{\partial f}{\partial L_x} \cdot (-1) = 0$$

Using calculus...

Using calculus... This is the order condition:

$$\frac{\partial u}{\partial L}(L, f(\bar{L} - L)) - \frac{\partial u}{\partial x}(L, f(\bar{L} - L)))f'(\bar{L} - L) = 0$$

$$\frac{\partial U}{\partial L} = \frac{\partial U}{\partial X} \frac{\partial f}{\partial L_x}$$

$$MRS = \frac{\frac{\partial U}{\partial L}}{\frac{\partial U}{\partial X}} = \frac{\partial f}{\partial L_x}$$

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$$\frac{\frac{\partial u}{\partial L}(L, f(\bar{L} - L))}{\frac{\partial u}{\partial x}(L, f(\bar{L} - L)))} = f'(\bar{L} - L)$$

$$\boxed{f'(\bar{L} - L)} = \frac{\frac{\partial u}{\partial L}(L, f(\bar{L} - L))}{\frac{\partial u}{\partial x}(L, f(\bar{L} - L))} = \boxed{MRS_{L,x}}$$

$f' > MRS_{L,x} \Rightarrow$ ROBINSON LE CONVENDRIA TRABAJAR MAS

$f' < MRS_{L,x} \Rightarrow$ ROBINO ↓ TRABAJO

$\boxed{f'_{L_x} = 2}$ $\boxed{MRS_{L,x} = 1}$

↓ 1 L (↑ 1 L_x)

→ 2+ COCOS

→ SE LOS DOY A ROBINSON

⇒ COMO $MRS_{L,x} = 1$

⇒ R ESTA + FELIZ.

- If Robinson gives up 1 unit of consumption in L, $f'(\bar{L} - L)$ describes how much more in terms of x Robinson will be able to consume

- If Robinson gives up 1 unit of consumption in L, $f'(\bar{L} - L)$ describes how much more in terms of x Robinson will be able to consume
- This is what is called a **Marginal Rate of Transformation** of good L to x

f'_{L_x}

$$f'(\bar{L} - L) = \frac{\frac{\partial u}{\partial L}(L, f(\bar{L} - L))}{\frac{\partial u}{\partial x}(L, f(\bar{L} - L))} = MRS_{L,x}$$

$$\boxed{f'(\bar{L}-L) = \frac{\frac{\partial u}{\partial L}(L, f(\bar{L}-L))}{\frac{\partial u}{\partial x}(L, f(\bar{L}-L))} = MRS_{L,x}}$$

$$\boxed{MRT_{L,x} = MRS_{L,x}.}$$

We can solve the above graphically.

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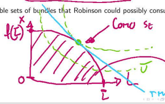
$$\boxed{\{(L,x) : x \leq f(L_X), L_x + L \leq \bar{L}\}}$$

describes the possible sets of bundles that Robinson could possibly consume in this economy.

Como se ve, depende de f



Q' es posible producir con la tecnologia de la economía

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This is called the **production possibilities set** (PPS)

We can solve the above graphically. The set

$$\{(L,x) : x \leq f(L_X), L_x + L \leq \bar{L}\}$$

describes the possible sets of bundles that Robinson could possibly consume in this economy.
This is called the **production possibilities set** (PPS)
The boundary of the PPS is the**production possibilities frontier** (PPF)

The frontier is basically described by the curve:

$$x = f(L_x), L_x \in [0, \bar{L}].$$

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$$x = f(L_x), L_x \in [0, \bar{L}].$$

The maximization problem for finding Pareto efficient allocations simply amounts to maximizing the utility of Robinson subject to being inside this constraint set.

Lecture 6: General Equilibrium

A few things I forgot to say about economies with production

Robinson Crusoe
- Econ 1 Intuition
- A concrete example

Two firms
- Graphical Approach
- Calculus Approach I
- Calculus Approach II
- Concrete Example

General Economies with Many Consumers and Production
- Solving the Maximization Problem

- Recall that at a Pareto optimum, we found that we must have $MRT_{L,x} = MRS_{L,x}$
- Suppose one is at an allocation where $MRT_{L,x} = 2 > MRS_{L,x} = 1$
- Such an allocation cannot be a Pareto efficient allocation. Why?

- Recall that at a Pareto optimum, we found that we must have $MRT_{L,x} = MRS_{L,x}$
- Suppose one is at an allocation where $MRT_{L,x} = 2 > MRS_{L,x} = 1$
- Such an allocation cannot be a Pareto efficient allocation. Why?
- One could potentially reorganize production to get an even better outcome for the consumer

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General Economies with Many Consumers and Production
- Solving the Maximization Problem

Suppose that the utility function is given by:

$$\boxed{u(x, y) = \sqrt{xy}}$$

Suppose there are ω_x units of x and suppose that the production is given by:

$$\boxed{f_y(x) = 2\sqrt{x}}$$

Then the maximization problem for Pareto efficiency is given by:

$$\max u(x, y) \text{ such that } y \leq f_y(x'), x' + x \leq \omega_x.$$

Then the maximization problem for Pareto efficiency is given by:

$$\max u(x, y) \text{ such that } y \leq f_y(x'), x' + x \leq \omega_x.$$

This simplifies to

$$\boxed{\max u(x, \underbrace{f_y(\omega_x - x)}_{Y}).} \longrightarrow \begin{array}{l} Y = f_Y(x') \\ x' + x = \omega_x \\ Y = f_Y(\omega_x - x) \end{array}$$

$$\text{MAX } \sqrt{X\, f_Y(\omega_X - X)}$$

$$\text{MAX } \sqrt{X\, 2\sqrt{\omega_X - X}}$$

We obtain the following FOC:

$$MRS_{x,y} = \frac{\frac{\partial u}{\partial x}(x^*, f_y(\omega_x - x^*))}{\frac{\partial u}{\partial y}(x^*, f_y(\omega_x - x^*))} = f_y'(\omega_x - x^*) = MRT_{x,y}.$$

Therefore,

$$\frac{2\sqrt{\omega_x - x^*}}{x^*} = \frac{1}{\sqrt{\omega_x - x^*}}.$$

Therefore,

$$\frac{2\sqrt{\omega_x - x^*}}{x^*} = \frac{1}{\sqrt{\omega_x - x^*}}.$$

This implies that

$$2(\omega_x - x^*) = x^* \Longrightarrow x^* = \frac{2\omega_x}{3}, y^* = 2\sqrt{\frac{\omega_x}{3}}$$

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General Economies with Many Consumers and Production

- The consumer has a utility function $u(x, y)$
- The consumer is endowed with 0 units of both x and y but x and y can be produced from labor and capital
- She is endowed with K units of capital and L units of labor
- There are two firms each of which produces a commodity x and y.
- Firm x produces x according to a production function $\boxed{f_x}$ and firm y produces y according to a production function $\boxed{f_y}$:

$$f_x(\ell_x, k_x), f_y(\ell_y, k_y).$$

To solve for the Pareto efficient allocation we solve:

$$\max_{x,y,\ell_x,\ell_y,k_x,k_y} u(x,y) \text{ such that } x \le f_x(\ell_x,k_x),\ y \le f_y(\ell_y,k_y),\ L \ge \ell_x+\ell_y,\ K \ge k_x+k_y.$$

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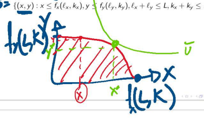
General Economies with Many Consumers and Production
- Solving the Maximization Problem

We look at the production possibilities set (PPS) of this economy:

$$PPS = \{(x,y) : x \le f_x(\ell_x,k_x),\ y \le f_y(\ell_y,k_y),\ \ell_x+\ell_y \le L,\ k_x+k_y \le K\}.$$ (x, y q' son factibles)



Then given the PPS, we want to maximize the utility of the agent subject to being inside the PPS. If we want to maximize the utility of the agent, we need:

1. The chosen (x^*, y^*) must be on the PPF.
2. The indifference curve of the consumer must be tangent to the PPF at (x^*, y^*).

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General Economies with Many Consumers and Production
- Solving the Maximization Problem

To find the PPF: given that x units of commodity x must be produced, what is the maximum amount of y's that can be produced?

$$\to \text{MAX } \underbrace{f_y(\ell_y,k_y)}_{y} \text{ s.a. } \left.\begin{array}{l} x \le f_x(\ell_x,k_x) \\ \ell_x+\ell_y \le L \\ k_x+k_y \le K \end{array}\right\} \left.\begin{array}{l} x = f_x(\ell_x,k_x) \\ \ell_x+\ell_y = L \\ k_x+k_y = K \end{array}\right\} \Rightarrow x = f_x(L-\ell_y, K-k_y)$$

To find the PPF: given that x units of commodity x must be produced, what is the maximum amount of y's that can be produced? Thus

$$\boxed{PPF(x) = \max_{\ell_y,k_y} f_y(\ell_y,k_y) \text{ such that } x = f_x(L-\ell_y, K-k_y).}$$

Setting up the Lagrangian we get:

$$\mathcal{L} = \max_{\ell_y,k_y} f_y(\ell_y,k_y) + \lambda(f_x(L-\ell_y, K-k_y) - x)$$

The first order conditions give us:

$$\frac{\partial \mathcal{L}}{\partial \ell_y} = \frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*) - \lambda\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_y} = \frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) - \lambda\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*) = 0$$

$$\Rightarrow \begin{array}{l} \dfrac{\partial f_y}{\partial \ell} = \lambda \dfrac{\partial f_x}{\partial \ell} \\ \dfrac{\partial f_y}{\partial K} = \lambda \dfrac{\partial f_x}{\partial K} \end{array} \Rightarrow \boxed{\frac{\frac{\partial f_y}{\partial \ell_y}}{\frac{\partial f_y}{\partial k_y}} = \frac{\frac{\partial f_x}{\partial \ell_x}}{\frac{\partial f_x}{\partial k_x}}}$$

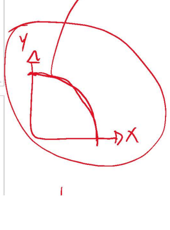
Unidades? $\dfrac{\text{Capital}}{\text{Trabajo}}$

$$\Rightarrow \boxed{TMST^y_{\ell,K} = TMST^x_{\ell,K}}$$

The first order conditions give us:

$$\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*) - \lambda\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*) = 0$$
$$\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) - \lambda\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*) = 0$$

$$\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*) = \lambda\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*)$$
$$\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) = \lambda\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*)$$



$$\boxed{2 = TMST^y_{\ell,K} > TMST^x_{\ell,K} = 1}$$

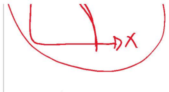
Firma X ↓1 ℓ
Firma Y ↑1ℓ, ↓1K.
X : ↑1K
⇒ X produce "="
Y produce "+"

X: ↓1ℓ, ↑2K.
Y: ↑1ℓ, ↓2K.
⇒ Y produce "="
X produce "+"

Thus at the optimum, we have:

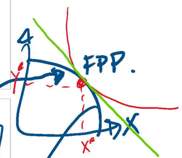
$$TRS^y_{\ell,k} = \frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)} = \frac{\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*)}{\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*)} = TRS^x_{\ell,k}.$$

⇒ X PRODUCE
Y PRODUCE



Thus at the optimum, we have:

$$\underline{TRS^{y}_{\ell,k}} = \frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)} = \frac{\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*)}{\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*)} = TRS^{x}_{\ell,k}.$$



- To actually solve for the optimal x^* and y^* we plug this back into the constraint $x = f_x(L-\ell_y^*, K-k_y^*)$
- Therefore at a Pareto optimum we must have $TRS^{y}_{\ell,k} = TRS^{x}_{\ell,k}$ equalized
- You should be able to come up with the Econ 1 intuition for this as we have done previously

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- A Pareto optimum also requires bullet point 2 above (i.e., indifference curve is tangent to PPF)
- The slope of the indifference curve is given by:

$$\boxed{-MRS_{x,y} = -\frac{\frac{\partial u}{\partial x}(x^*, y^*)}{\frac{\partial u}{\partial y}(x^*, y^*)}.}$$

- What is the slope of the PPF? $\boxed{Y = PPF(X)}$

- Note that mathematically, this is given by $PPF'(x)$
- How do we calculate that?

$$PPF(x) = \boxed{\max_{\ell_y,k_y} f_y(\ell_y, k_y) + \lambda(f_x(L-\ell_y, K-k_y) - x))}$$

- By the envelope theorem

$$\begin{aligned} PPF'(x) &= -\lambda \\ &= \boxed{-\frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*)}} \\ &= -\frac{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*)} \\ &= -MRT_{x,y} \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_y} = \frac{\partial f_y}{\partial \ell_y} - \lambda \frac{\partial f_x}{\partial \ell_x} = 0$$

$$-\lambda = -\frac{\partial f_y / \partial \ell_y}{\partial f_x / \partial \ell_x}$$

UNIDADES? $\frac{Y}{X}$

→ COMO LA ECONOMIA SUSTITUYE PRODUCCIÓN DE X POR PRODUCCIÓN DE Y

- Note that mathematically, this is given by $PPF'(x)$
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$$PPF(x) = \max_{\ell_y,k_y} f_y(\ell_y, k_y) + \lambda(f_x(L-\ell_y, K-k_y) - x))$$

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$$\begin{aligned} PPF'(x) &= -\lambda \\ &= -\frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial \ell}(L-\ell_y^*, K-k_y^*)} \\ &= -\frac{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial k}(L-\ell_y^*, K-k_y^*)} \Rightarrow \\ &= -MRT_{x,y} \end{aligned}$$

- Therefore, at a Pareto optimum:

$$\boxed{MRT_{x,y} = MRS_{x,y}.}$$

Thus we have learned the following: A Pareto efficient allocation is characterized by two conditions:

1. (x^*, y^*) is on the PPF: $\boxed{TRS^{x}_{\ell,k} = TRS^{y}_{\ell,k}}$.
2. At (x^*, y^*) the indifference curve is tangent to the PPF: $MRS_{x,y} = MRT_{x,y}$.

TASA MARGINAL TRANSFORMACIÓN

→ PRODUCCIÓN

→ CONSUMO

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General Economies with Many Consumers and Production
Solving the Maximization Problem

We solve directly the original maximization problem.

$$\max_{x,y,\ell_x,\ell_y,k_x,k_y} u(x,y) \text{ such that } x \le f_x(\ell_x, k_x), y \le f_y(\ell_y, k_y), \\ L \ge \ell_x + \ell_y, K \ge k_x + k_y.$$

$$\Rightarrow \left. \begin{aligned} x &= f_x(\ell_x, k_x) \\ y &= f_y(\ell_y, k_y) \\ L &= \ell_x + \ell_y \\ K &= k_x + k_y \end{aligned} \right\} \quad \begin{aligned} x &= f_x(\ell_x, k_x) \\ y &= f_y(L - \ell_x, K - k_x) \end{aligned}$$

$$\text{MAX } u(f_x(\ell_x, k_x), f_y(L-\ell_x, K-k_x))$$

We can simplify the problem:

$$\boxed{\max_{\ell_x,k_x} u(f_x(\ell_x, k_x), f_y(L-\ell_x, K-k_x))}$$

Then the first order conditions give us:

$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*) = 0,$$
$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial k}(L-\ell_x^*, K-k_x^*) = 0$$

$f_y(\ell_y, k_y)$

$$\frac{\partial u}{\partial x}\frac{\partial f_x}{\partial \ell_x} = \frac{\partial u}{\partial y}\cdot\frac{\partial f_y}{\partial \ell_y} \Rightarrow \frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial f_y/\partial \ell_y}{\partial f_x/\partial \ell_x}$$

$$\boxed{MRS_{x,y} = MRT_{x,y}}$$

$$\frac{\partial u}{\partial x}\frac{\partial f_x}{\partial k_x} = \frac{\partial u}{\partial y}\frac{\partial f_y}{\partial k_y}$$

$$\boxed{\frac{\partial f_x/\partial \ell_x}{\partial f_x/\partial k_x} = \frac{\partial f_y/\partial \ell_y}{\partial f_y/\partial k_y}}$$

$$TMST^{x}_{\ell,K} = TMST^{y}_{\ell,K}$$
$$TRS^{x}_{\ell,K} = TRS^{y}_{\ell,K}$$

Then the first order conditions give us:

$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*) = 0,$$
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$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) = \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*),$$

$TMST^x_{\ell,k} = TMST^y_{\ell,k}$
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$$\frac{\partial u}{\partial x}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*,k_x^*)-\frac{\partial u}{\partial y}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_y}{\partial k}(L-\ell_x^*,K-k_x^*)=0$$

$$\frac{\partial u}{\partial x}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*,k_x^*)=\frac{\partial u}{\partial y}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*,K-k_x^*),$$
$$\frac{\partial u}{\partial x}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*,k_x^*)=\frac{\partial u}{\partial y}(f_x(\ell_x^*,k_x^*),f_y(L-\ell_x^*,K-k_x^*))\frac{\partial f_y}{\partial k}(L-\ell_x^*,K-k_x^*).$$

We obtain:

$$\boxed{\begin{aligned} TRS^x_{\ell,k} &= TRS^y_{\ell,k}, \\ MRS_{x,y} &= MRT_{x,y}, \end{aligned}}$$

Lecture 6: General Equilibrium

A few things I forgot to say about economies with production

Robinson Crusoe
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General Economies with Many Consumers and Production
- Solving the Maximization Problem

Suppose that the utility function are given by:

$$\boxed{u(x,y)=\sqrt{xy}}$$

and suppose that the production functions are given by:

$$f_x(\ell_x,k_x)=\sqrt{\ell_x k_x}, f_y(\ell_y,k_y)=\sqrt{\ell_y k_y}.$$

(L, K) DOTACIÓN INICIAL.

Then Pareto efficiency involves solving the following maximization problem:

$$\max \sqrt{xy} \text{ such that } x=f_x(\ell_x,k_x), y=f_y=f_y(\ell_y,k_y).$$

$\ell x + \ell y = L$
$Kx + Ky = K.$

Approach 1: First lets characterize the PPF.

$\boxed{\text{MAX } y \text{ s.a. FACTIBLE}}$

$$PPF(x)=\max_{\ell x, kx} f_y(L-\ell_x,K-k_x) \text{ such that } f_x(\ell_x,k_x)=x.$$

$$\mathcal{L} = f_y(L-\ell_x, K-k_x) + \lambda(x - f_x(\ell_x, k_x))$$

$$\frac{\partial \mathcal{L}}{\partial \ell x} = \frac{\partial f_y}{\partial \ell y}\cdot(-1) - \lambda\frac{\partial f_x}{\partial \ell x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial Kx} = \frac{\partial f_y}{\partial K y}(-1) - \lambda\frac{\partial f_x}{\partial K x} = 0$$

$$\Rightarrow \frac{\dfrac{\partial f_y}{\partial \ell y}}{\dfrac{\partial f_y}{\partial K y}} = \frac{\dfrac{\partial f_x}{\partial \ell x}}{\dfrac{\partial f_x}{\partial K x}}$$

By the first order condition, we need:

$$\frac{k_x^*}{\ell_x^*}=\frac{\frac{\partial f_x}{\partial \ell}(\ell_x^*,k_x^*)}{\frac{\partial f_x}{\partial k}(\ell_x^*,k_x^*)}=\frac{\frac{\partial f_y}{\partial \ell}(L-\ell_x^*,K-k_x^*)}{\frac{\partial f_y}{\partial k}(L-\ell_x^*,K-k_x^*)}=\frac{K-k_x^*}{L-\ell_x^*}\Longrightarrow k_x^*=\frac{K}{L}\ell_x^*.$$

$$\mathcal{L} = \sqrt{(L-\ell x)(K-Kx)} + \lambda(x - \sqrt{\ell x Kx})$$

$$\frac{\partial \mathcal{L}}{\partial \ell x} = \frac{1}{2}\sqrt{\frac{(K-Kx)}{(L-\ell x)}}(-1) - \lambda\frac{1}{2}\sqrt{\frac{Kx}{\ell x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial Kx} = \frac{1}{2}\sqrt{\frac{(L-\ell x)}{(K-Kx)}}(-1) - \lambda\frac{1}{2}\sqrt{\frac{\ell x}{Kx}} = 0$$

$$\Rightarrow \frac{(K-Kx)}{L-Lx} = \frac{Kx}{\ell x} \Rightarrow K\ell x - Kx\ell x = KxL - Kx\ell x \Rightarrow K\ell x = KxL$$

$$\boxed{\frac{K}{L} = \frac{Kx}{\ell x}}$$

Plug this back into the constraint:

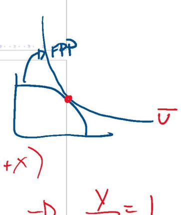
$$f_x(\ell_x^*,k_x^*)=x\Longrightarrow\sqrt{\ell_x k_x}=x\Longrightarrow\ell_x^*=\sqrt{\frac{L}{K}}x.$$

$$x = f_x(Lx, Kx) = \sqrt{LxKx} \qquad Kx = \frac{K}{L}\ell x$$

$$x = \sqrt{\ell x^2\frac{K}{L}} \Rightarrow x = \ell x\sqrt{\frac{K}{L}} \Rightarrow \boxed{x\sqrt{\frac{L}{K}} = \ell x}.$$

Therefore

$$f_y(L-\ell_x,K-k_x)=y$$
$$\boxed{\sqrt{\left(L-\sqrt{\frac{L}{K}}x\right)\left(K-\sqrt{\frac{K}{L}}x\right)}=y}$$
$$PPF(x)=\left(\sqrt{KL}-x\right)$$



Then we need to maximize the following:

$$\max_{x,y}\sqrt{xy} \text{ such that } \boxed{y=\sqrt{KL}-x.}$$

$$\mathcal{L} = \sqrt{xy} + \lambda(y - \sqrt{KL} + x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}} + \lambda = 0 \Rightarrow \frac{y}{x} = 1$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}} + \lambda = 0$$

$$\boxed{y = x}$$

$$y = \sqrt{KL} - x$$

$$2y = \sqrt{KL}$$

The Pareto efficient allocation is given by:

$$\left(x^*=y^*=\frac{1}{2}\sqrt{KL},\ell_x^*=\ell_y^*=\frac{1}{2}L,k_x^*=k_y^*=\frac{1}{2}K\right).$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}\sqrt{\frac{L}{y}} + \lambda = 0$$

The Pareto efficient allocation is given by:

$$\left(x^* = y^* = \frac{1}{2}\sqrt{KL}, \ell_x^* = \ell_y^* = \frac{1}{2}L, k_x^* = k_y^* = \frac{1}{2}K\right).$$

$y = x$

$y = \sqrt{KL} - x$

$2y = \sqrt{KL}$

$\boxed{y = \frac{1}{2}\sqrt{KL} = x}$

$$x = \sqrt{\ell_x^2 \frac{K}{L}} \Rightarrow x = \ell_x\sqrt{\frac{K}{L}} \Rightarrow \boxed{x\sqrt{\frac{L}{K}} = \ell_x}$$

$$\boxed{x\sqrt{\frac{K}{L}} = k_x}$$

$$y = \sqrt{(L-\ell_x)(K-k_x)}$$

$$y = \sqrt{\left(L - x\sqrt{L/K}\right)\left(K - x\sqrt{K/L}\right)} \Rightarrow y \text{ como función } x$$

$\Rightarrow$ FPP

Lecture 6: General Equilibrium

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Lecture 6: General Equilibrium

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General Economies with Many Consumers and Production

The set of Pareto efficient allocations will be characterized by the following maximization problem:

$$\max_{(x,z)} u_1(x_1^1, x_2^1, \ldots, x_L^1) \text{ such that } u_2(x_1^2, \ldots, x_L^2) \geq \underline{u}_2 = u_2(\hat{x}_1^2, \ldots, \hat{x}_L^2),$$

$$\vdots$$

$$u_I(x_1^I, \ldots, x_L^I) \geq \underline{u}_I = u_I(\hat{x}_1^I, \ldots, \hat{x}_L^I),$$

Q' TODOS LOS DEMAS ESTEN MINIMO EN UN NIVEL DADO

$$x_1^1 + \cdots x_1^I + z_1^1 + \cdots + z_1^J \leq \sum_{j:\ell(j)=1} f_j(z_1^j, \ldots, z_L^j) + \sum_{i=1}^{I} \omega_1^i,$$

$$\vdots$$

$$x_L^1 + \cdots x_L^I + z_L^1 + \cdots + z_L^J \leq \sum_{j:\ell(j)=L} f_j(z_1^j, \ldots, z_L^j) + \sum_{i=1}^{I} \omega_L^i.$$

FACTIBLE.

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~~$MRS^1_{a,b} = MRS^2_{b,c}$~~

Theorem

Suppose that utility functions are strictly monotone, differentiable, and quasi-concave. Suppose also that $(\hat{x}, \hat{z})$ is an interior allocation. Then $(\hat{x}, \hat{z})$ is Pareto efficient if and only if all of the following hold:

1. *For every $\ell \neq \ell'$, marginal rates of substitution of any pair of commodities are equalized across consumers:*

$$\frac{\frac{\partial u_1}{\partial x_\ell}(\hat{x}_1^1, \ldots, \hat{x}_L^1)}{\frac{\partial u_1}{\partial x_{\ell'}}(\hat{x}_1^1, \ldots, \hat{x}_L^1)} = \frac{\frac{\partial u_2}{\partial x_\ell}(\hat{x}_1^2, \ldots, \hat{x}_L^2)}{\frac{\partial u_2}{\partial x_{\ell'}}(\hat{x}_1^2, \ldots, \hat{x}_L^2)} = \cdots = \frac{\frac{\partial u_I}{\partial x_\ell}(\hat{x}_1^I, \ldots, \hat{x}_L^I)}{\frac{\partial u_I}{\partial x_{\ell'}}(\hat{x}_1^I, \ldots, \hat{x}_L^I)}.$$

$\Rightarrow MRS^1_{\ell,\ell'} = MRS^2_{\ell,\ell'} = \ldots = MRS^I_{\ell,\ell'}$

2. *For every $\ell \neq \ell'$, technical rates of substitution of inputs ℓ and ℓ' are equalized across firms:*

$$\frac{\frac{\partial f_1}{\partial z_\ell}(\hat{z}_1^1, \ldots, \hat{z}_L^1)}{\frac{\partial f_1}{\partial z_{\ell'}}(\hat{z}_1^1, \ldots, \hat{z}_L^1)} = \frac{\frac{\partial f_2}{\partial z_\ell}(\hat{z}_1^2, \ldots, \hat{z}_L^2)}{\frac{\partial f_2}{\partial z_{\ell'}}(\hat{z}_1^2, \ldots, \hat{z}_L^2)} = \cdots = \frac{\frac{\partial f_J}{\partial z_\ell}(\hat{z}_1^J, \ldots, \hat{z}_L^J)}{\frac{\partial f_J}{\partial z_{\ell'}}(\hat{z}_1^J, \ldots, \hat{z}_L^J)}.$$

$\Rightarrow TMST^1_{\ell,\ell'} = TMST^2_{\ell,\ell'} = \ldots = TMST^J_{\ell,\ell'}$

~~$TMST^1_{a,b} = TMST^2_{b,c}$~~

3. *For every $\ell \neq \ell'$, the marginal rates of transformation is equal to the marginal rates of substitution:*

$$\frac{\frac{\partial f_j}{\partial z_{\ell''}}(\hat{z}_1^j, \ldots, \hat{z}_L^j)}{\frac{\partial f_{j'}}{\partial z_{\ell''}}(\hat{z}_1^{j'}, \ldots, \hat{z}_L^{j'})} = \frac{\frac{\partial u_1}{\partial x_\ell}(\hat{x}_1^1, \ldots, \hat{x}_L^1)}{\frac{\partial u_1}{\partial x_{\ell'}}(\hat{x}_1^1, \ldots, \hat{x}_L^1)} = \cdots = \frac{\frac{\partial u_I}{\partial x_\ell}(\hat{x}_1^I, \ldots, \hat{x}_L^I)}{\frac{\partial u_I}{\partial x_{\ell'}}(\hat{x}_1^I, \ldots, \hat{x}_L^I)}.$$

$\Rightarrow \boxed{MRS^i_{\ell,\ell'} = MRT_{\ell,\ell'}}$

O. P.