

Lecture 12: Game Theory // Nash equilibrium

Mauricio Romero

Lecture 12: Game Theory // Nash equilibrium

Dominance

Nash equilibrium

Some examples

Relationship to dominance

Examples

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Beauty contest

- ▶ Consider the following game among 100 people. Each individual selects a number, s_i , between 20 and 60.
- ▶ Let a_{-i} be the average of the number selected by the other 99 people. i.e.
$$a_{-i} = \sum_{j \neq i} \frac{s_j}{99}.$$
- ▶ The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}a_{-i})^2$

Beauty contest

- ▶ Each individual maximizes his utility, FOC:

$$-2\left(s_i - \frac{3}{2}a_{-i}\right) = 0$$

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- ▶ but $a_{-i} \in [20, 60]$

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- ▶ Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- ▶ That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- ▶ but $a_{-i} \in [20, 60]$
- ▶ Therefore $s_i = 20$ is dominated by $s_i = 30$

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- ▶ Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$)
- ▶ Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45

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- ▶ Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45
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- ▶ 60 would dominate any other selection and therefore all the players select 60.

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- ▶ Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$)
- ▶ 60 would dominate any other selection and therefore all the players select 60.
- ▶ The solution by means of iterated elimination of dominated strategies is
 $(\underbrace{60, 60, \dots, 60}_{100 \text{ times}})$

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Dominance

Weakly dominated strategies

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Cournot Competition

Cartels

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- ▶ There is no strictly dominated strategy

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- ▶ It's tempting to think player 1 would never play *C*

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- ▶ There is no strictly dominated strategy
- ▶ However, C always gives at least the same utility to player 1 as B
- ▶ It's tempting to think player 1 would never play C
- ▶ However, if player 1 is sure that player two is going to play a he would be completely indifferent between playing B or C

Definition

s_i weakly dominates s'_i if for all opponent pure strategy profiles, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

and there is at least one opponent strategy profile $s''_{-i} \in S_{-i}$ for which

$$u_i(s_i, s''_{-i}) > u_i(s'_i, s''_{-i}).$$

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- ▶ Rationality is not enough
- ▶ Even so, it sounds “logical” to do so and has the potential to greatly simplify a game
- ▶ There is a problem, and that is that the order in which we eliminate the strategies matters

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- ▶ If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A . This leads to the result (C, a) .

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- ▶ If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A . This leads to the result (C, a) .
- ▶ If on the other hand, we notice that A is also weakly dominated by C then we can eliminate it in the first round, and this would eliminate a in the second round and therefore B would be eliminated. This would result in (C, b) .

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Remember the definition of competitive equilibrium in a market economy.

Definition

A competitive equilibrium in a market economy is a vector of prices and baskets x_i such that: 1) x_i maximizes the utility of each individual given the price vector i.e.

$$x_i = \arg \max_{p \cdot x_i \leq p \cdot w_i} u(x_i)$$

2) the markets empty.

$$\sum_i x_i = \sum_i w_i$$

- ▶ 1) means that given the prices, individuals have no incentive to demand a different amount

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- ▶ The idea is to extend this concept to strategic situations

Best response

We denote $BR_i(s_{-i})$ (best response) as the set of strategies of individual i that maximize her utility given that other individuals follow the strategy profile s_{-i} .
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Given a strategy profile of opponents s_{-i} , we can define the best response of player i :

$$BR_i(s_{-i}) = \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}).$$

- ▶ $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$

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- ▶ $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$
- ▶ There could be multiple strategies in $BR_i(s_{-i})$ but all such strategies give the same utility to player i if the opponents are indeed playing according to s_{-i}

Nash equilibrium

Definition

Suppose that we have a game $(I = \{1, 2, \dots, n\}, S_1, \dots, S_n, u_1, \dots, u_n)$. Then a strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a pure strategy **Nash equilibrium** if for every i and for every $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

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- ▶ Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- ▶ once this equilibrium is reached, nobody has incentives to move from there
- ▶ This is a concept of stability, but there is no way to ensure, or predict, that the game will reach this equilibrium

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- ▶ Let s_{-i} be the number selected by the other individual.
- ▶ The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}s_{-i})^2$

Beauty contest

The best response of an individual is given by

$$s_i(s_{-i})^* = \begin{cases} \frac{3}{2}s_{-i} & \text{if } s_{-i} \leq 40 \\ 60 & \text{if } s_{-i} > 40 \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)

Prisoner's dilemma

	C	NC
C	5,5	0,10
NC	10,0	2,2

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The best response functions are:

$$BR_i(s_{-i}) = \begin{cases} NC & \text{if } s_{-i} = C \\ C & \text{if } s_{-i} = NC \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play NC, i.e., (NC, NC))

Prisoner's dilemma – A trick

Best response of 1 to 2 playing C

	C	NC
C	5,5	0,10
NC	<u>10,0</u>	2,2

Prisoner's dilemma – A trick

Best response of 1 to 2 playing NC

	C	NC
C	5,5	0,10
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Prisoner's dilemma – A trick

Best response of 2 to 1 playing C

	C	NC
C	5,5	0, <u>10</u>
NC	<u>10</u> ,0	<u>2</u> , <u>2</u>

Prisoner's dilemma – A trick

Best response of 2 to 1 playing NC

	C	NC
C	5,5	0, <u>10</u>
NC	<u>10</u> ,0	<u>2</u> , <u>2</u>

When underlined for both players, it is a Nash equilibrium (both are doing their BR)

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

Battle of the sexes

	G	P
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$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Battle of the sexes

	G	P
G	<u>2</u> , <u>1</u>	0,0
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$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Thus, (G, G) y (P, P) are both Nash equilibrium

Matching pennies (Pares o Nones) – Simultaneous

	1	2
1	(1000,-1000)	(-1000,1000)
2	(-1000,1000)	(1000,-1000)

Matching pennies (Pares o Nones) – Simultaneous

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Matching pennies (Pares o Nones) – Simultaneous

	1	2
1	(<u>1000</u> , -1000)	(-1000, <u>1000</u>)
2	(-1000, <u>1000</u>)	(<u>1000</u> , -1000)

$$BR_1(s_2) = \begin{cases} 1 & \text{if } s_2 = 1 \\ 2 & \text{if } s_2 = 2 \end{cases}$$

$$BR_2(s_1) = \begin{cases} 2 & \text{if } s_1 = 1 \\ 1 & \text{if } s_1 = 2 \end{cases}$$

There is no Nash equilibrium in pure strategies

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Nash equilibrium survive IDSDS

Theorem

Every Nash equilibrium survives the iterative elimination of strictly dominated strategies

Proof

By contradiction:

- ▶ Suppose it is not true

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- ▶ It must have been that

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- ▶ In particular

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- ▶ But this means s_i^* is not the best response of individual i to s_{-i}^*
- ▶ And this is a contradiction!

Nash equilibrium survive IDSDS

Theorem

If the process of IDSDS comes to a single solution, that solution is a Nash Equilibrium and is unique.

Proof

First let's prove it's a Nash Equilibrium. The fact that it is unique is trivial by the previous theorem.

Proof.

By contradiction:

- ▶ Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium



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- ▶ Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- ▶ For some individual i there exists s_i such that

$$u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$



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- ▶ But then s_i could not have been eliminated



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By contradiction:

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- ▶ Suppose that there are two firms that produce the same product have zero marginal cost of production.
- ▶ If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

Cournot Competition

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- ▶ Suppose that there are two firms that produce the same product have zero marginal cost of production.
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$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

- ▶ Strategy space is $S_i = [0, +\infty)$
- ▶ The utility function of player i is given by:

$$\begin{aligned}\pi_1(q_1, q_2) &= (120 - (q_1 + q_2))q_1, \\ \pi_2(q_1, q_2) &= (120 - (q_1 + q_2))q_2.\end{aligned}$$

Cournot Competition

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Cournot Competition

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- ▶ Are there any strictly dominated strategies?
- ▶ The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- ▶ Are there any others? given q_{-i} ,

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

Cournot Competition

- ▶ Are there any strictly dominant strategies? The answer is no, why?
- ▶ Are there any strictly dominated strategies?
- ▶ The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- ▶ Are there any others? given q_{-i} ,

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

- ▶ Therefore 60 strictly dominates any $q_i \in (60, 120]$

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

- ▶ for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

- ▶ for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$
- ▶ Such a q_i can never be strictly dominated

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

- ▶ for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$
- ▶ Such a q_i can never be strictly dominated
- ▶ After one round of deletion of strictly dominated strategies, we are left with:
 $S_i = [0, 60]$

Cournot Competition



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Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

▶ $q_{-i} = [0, 60]$

▶ Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

▶ $q_{-i} = [0, 60]$

▶ Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$

▶ After two rounds of deletion of strictly dominated strategies, we are left with:
 $S_i = [30, 60]$

Cournot Competition

- ▶
$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- ▶ $q_{-i} = [30, 60]$
- ▶ 45 strictly dominates all strategies $q_i \in (45, 60]$
- ▶ After three rounds of deletion of strictly dominated strategies, we are left with:
 $S_i = [30, 45]$

Cournot Competition



$$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

▶ $q_{-i} = [30, 45]$

▶ 37.5 strictly dominates all strategies $q_i \in [30, 37.5]$

▶ After four rounds of deletion of strictly dominated strategies, we are left with:
 $S_i = [37.5, 45]$

Cournot Competition

- ▶ After (infinitely) many iterations, the only remaining strategies are $S_i = 40$
- ▶ The unique solution by IDSDS is $q_1^* = q_2^* = 40$.

Cournot Competition

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$$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$

- ▶ We can solve for q_1^* and q_2^* to obtain:

$$q_1^* = 40, q_2^* = 40, Q^* = 80, \Pi_1^* = \Pi_2^* = 1600.$$

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- ▶ In a duopoly, externalities are imposed on the other firm

Lecture 12: Game Theory // Nash equilibrium

Dominance

Weakly dominated strategies

Nash equilibrium

Some examples

Relationship to dominance

Examples

Cournot Competition

Cartels

Cartels

- ▶ Suppose there are three firms who face zero marginal cost
- ▶ The inverse demand function is given by:

$$p(q_1 + q_2 + q_3) = 1 - q_1 - q_2 - q_3 = 1 - Q$$

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- ▶ In a Nash equilibrium we must have:

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- ▶ $q_1^* = q_2^* = q_3^* = \frac{1}{4}$

- ▶ Price is $p^* = 1/4$ and all firms get the same profits of $1/16$

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- ▶ Therefore

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- ▶ Firms 1 and 2 suffered, while firm 3 is better off!
- ▶ Firm 3 is obtaining a disproportionate share of the joint profits (more than $1/3$)

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- ▶ Firm 3 clearly wants to stay out

Cartels

There are many difficulties associated with sustaining collusive agreements (e.g., the OPEC cartel)