Lecture12.pdf

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Lecture12....

Lecture 12: Game Theory // Nash equilibrium

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Lecture 12: Game Theory // Nash equilibrium

Lecture 12: Game Theory // Nash equilibrium

Beauty contest

- Consider the following game among 100 people. Each individual selects a number, s_i, between 20 and 60.
- Let a_{-i} be the average of the number selected by the other 99 people. i.e. $a_{-i} = \sum_{j \neq i} \frac{s_j}{99}$.
- The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}a_{-i})^2$

Beauty contest

- Each individual maximizes his utility, FOC:

$$\boxed{-2(s_i - \frac{3}{2}a_{-i}) = 0}$$

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- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others

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- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- That is they would like to choose $\boxed{s_i = \frac{3}{2}a_{-i}}$
- but $a_{-i} \in [20, 60]$

$\frac{3}{2}a_{-i} \in [\frac{3}{2} \cdot 20, \frac{3}{2} 60]$

$\in [30, 90]$

Beauty contest

- Each individual maximizes his utility, FOC:

$$-2(s_i - \frac{3}{2}a_{-i}) = 0$$

- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- but $a_{-i} \in [20, 60]$
- Therefore $s_i = 20$ is dominated by $s_i = 30$

$s_i \in [20,30)$ ESTA DOMINADO POR $s_i = 30$

Beauty contest

- The same goes for any number between 20 (inclusive) and 30 (not included)

$a_{-i} \in [30, 60]$

$\frac{3}{2}a_{-i} \in [45, 90]$

$s_i = 45$ DOMINA $s_i \in [30, 45)$

Beauty contest

- The same goes for any number between 20 (inclusive) and 30 (not included)
- Knowing this, all individuals believe that everyone else will select a number between 30 and 60 (i.e., $a_{-i} \in [30, 60]$)

$s_{-i} \in [45, 60]$

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$\frac{3}{2}a_{-i} [67.5, 90]$

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- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45

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- 60 would dominate any other selection and therefore all the players select 60.

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- Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$)
- 60 would dominate any other selection and therefore all the players select 60.
- The solution by means of iterated elimination of dominated strategies is $\underbrace{(60, 60, ..., 60)}_{\text{100 times}}$

Lecture 12: Game Theory // Nash equilibrium

Dominance
- Weakly dominated strategies

Nash equilibrium

Some examples

Relationship to dominance

Examples
- Cournot Competition
- Cartels

S2

S1

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

$C \succcurlyeq B \rightarrow a \succcurlyeq b \rightarrow C \succcurlyeq A \rightarrow (C, a)$

$C \succcurlyeq A \rightarrow b \succcurlyeq a. \rightarrow C \succcurlyeq B \rightarrow (C, b)$

- There is no strictly dominated strategy

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- It's tempting to think player 1 would never play C

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- There is no strictly dominated strategy
- However, C always gives at least the same utility to player 1 as B
- It's tempting to think player 1 would never play C
- However, if player 1 is sure that player two is going to play a he would be completely indifferent between playing B or C

Definition

s_i weakly dominates s_i' if for all opponent pure strategy profiles, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

and there is at least one opponent strategy profile $s''_{-i} \in S_{-i}$ for which

$$u_i(s_i, s''_{-i}) > u_i(s_i', s''_{-i}).$$

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- Given the assumptions we have, we can not eliminate a weakly dominated strategy
- Rationality is not enough
- Even so, it sounds "logical" to do so and has the potential to greatly simplify a game
- There is a problem, and that is that the order in which we eliminate the strategies matters

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	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).

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	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).
- If on the other hand, we notice that A is also weakly dominated by C then we can eliminate it in the first round, and this would eliminate a in the second round and therefore B would be eliminated. This would result in (C, b).

Remember the definition of competitive equilibrium in a market economy.

Definition
A competitive equilibrium in a market economy is a vector of prices and baskets x_i such that: 1) x_i maximizes the utility of each individual given the price vector i.e.

$$x_i = \arg \max_{p \cdot x_i \leq p \cdot w_i} u(x_i)$$

2) the markets empty.

$$\sum_i x_i = \sum_i w_i$$

- 1) means that given the prices, individuals have no incentive to demand a different amount

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- 1) means that given the prices, individuals have no incentive to demand a different amount
- The idea is to extend this concept to strategic situations

Best response

We denote $BR_i(s_{-i})$ (best response) as the set of strategies of individual i that maximize her utility given that other individuals follow the strategy profile s_{-i}. Formally,

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Given a strategy profile of opponents s_{-i}, we can define the best response of player i:

$$BR_i(s_{-i}) = \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}).$$

- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$

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- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$
- There could be multiple strategies in $BR_i(s_{-i})$ but all such strategies give the same utility to player i if the opponents are indeed playing according to s_{-i}

Nash equilibrium

Definition
Suppose that we have a game $(I = \{1, 2, \dots n\}, S_1, \dots, S_n, u_1, \dots, u_n)$. Then a strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a pure strategy **Nash equilibrium** if for every i and for every $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*). \rightsquigarrow s_i^* \in MR_i(s_{-i}^*)$$

Nash equilibrium

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- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate

Nash equilibrium

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- once this equilibrium is reached, nobody has incentives to move from there

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- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- once this equilibrium is reached, nobody has incentives to move from there
- This is a concept of stability, but there is no way to ensure, or predict, that the game will reach this equilibrium

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Beauty contest

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- Let s_{-i} be the number selected by the other individual.
- The utility function of the individual i is $u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}s_{-i})^2$

$$\frac{\partial u_i(s_i; s_{-i})}{\partial s_i} = -2\left(s_i - \frac{3}{2}s_{-i}\right) = 0$$

$$s_i = \frac{3}{2}s_{-i}$$

$$\boxed{MR_i(s_{-i}) = \frac{3}{2}s_{-i}}$$

$$MR_1(s_2) = \frac{3}{2}s_2 \rightarrow MR_1(s_2) = \begin{cases} \frac{3}{2}s_2 & \text{si } s_2 \in [20, 40] \\ 60 & \text{si } s_2 \geq 40 \end{cases}$$

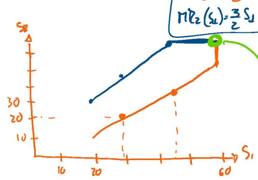
$$MR_2(s_1) = \frac{3}{2}s_1 \rightarrow MR_2(s_1) = \begin{cases} \frac{3}{2}s_1 & \text{si } s_1 \in [20, 40] \\ 60 & \text{si } s_1 \geq 40 \end{cases}$$

Beauty contest

The best response of an individual is given by

$$s_i(s_{-i})^* = \begin{cases} \frac{3}{2}s_{-i} & \text{if } s_{-i} \leq 40 \\ 60 & \text{if } s_{-i} > 40 \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)



E. N.
$(s_1 = 60, s_2 = 60)$

Prisoner's dilemma

	C	NC
C	5,5	0,10
NC	10,0	2,2

$$MR_1(s_2) = \begin{cases} NC & s_2 = C \\ NC & s_2 = NC \end{cases}$$

$$MR_2(s_1) = NC$$

$EN = (NC, NC)$

$\rightarrow$ ~~(C, NC)~~

(NC, C)

Prisoner's dilemma

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$\rightarrow EN = (NC, NC)$

The best response functions are:

$$BR_i(s_{-i}) = \begin{cases} NC & \text{if } s_{-i} = C \\ NC & \text{if } s_{-i} = NC \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play NC, i.e., (NC, NC))

Prisoner's dilemma – A trick

Best response of 1 to 2 playing C

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 1 to 2 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing C

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

When underlined for both players, it is a Nash equilibrium (both are doing their BR)

Battle of the sexes

S_2

S_1

	G	P
G	2,1	0,0
P	0,0	1,2

$$MR_1(S_2) = \begin{cases} G & S_2 = G \\ P & S_2 = P \end{cases}$$

$$MR_2(S_1) = \begin{cases} G & S_1 = G \\ P & S_1 = P \end{cases}$$

EN: $\{(G,G); (P,P)\}$

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Battle of the sexes

	G	P
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$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Thus, (G,G) y (P,P) are both Nash equilibrium

Matching pennies (Pares o Nones) – Simultaneous

	1	2
1	(1000,-1000)	(-1000,1000)
2	(-1000,1000)	(1000,-1000)

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$$BR_1(s_2) = \begin{cases} 1 & \text{if } s_2 = 1 \\ 2 & \text{if } s_2 = 2 \end{cases}$$

$$BR_2(s_1) = \begin{cases} 2 & \text{if } s_1 = 1 \\ 1 & \text{if } s_2 = 2 \end{cases}$$

There is no Nash equilibrium in pure strategies

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Nash equilibrium survive IDSDS

Theorem
Every Nash equilibrium survives the iterative elimination of strictly dominated strategies

Proof

By contraction:

- Suppose it is not true

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- Without loss of generality say we eliminated the strategy s_i^* of individual i

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- Lets zoom in in the round where we first eliminate a strategy that is part of s^*
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- It must have been that

$$u_i(s_i^*, s_{-i}) < u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

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- In particular

$$u_i(s_i^*, s_{-i}*) < u_i(s_i, s_{-i}^*)$$

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- But this means s_i^* is not the best response of individual i to s_{-i}^*

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$$u_i(s_i^*, s_{-i}*) < u_i(s_i, s_{-i}^*)$$

- But this means s_i^* is not the best response of individual i to s_{-i}^*
- And this is a contradiction!

Nash equilibrium survive IDSDS

Theorem
If the process of IDSDS comes to a single solution, that solution is a Nash Equilibrium and is unique.

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium

□

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

$$u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$

□

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Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

$$u_i(s_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i})$$

- But then s_i could not have been eliminated

□

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By contradiction:

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Dominance

Nash equilibrium

Some examples

Relationship to dominance

Examples

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Lecture 12: Game Theory // Nash equilibrium

Dominance
  Weakly dominated strategies

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Relationship to dominance

Examples
  Cournot Competition
  Cartels

Cournot Competition

- We will apply the concept of pure Nash equilibrium to analyze oligopoly markets

Cournot Competition

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Cournot Competition

- We will apply the concept of pure Nash equilibrium to analyze oligopoly markets
- Suppose that there are two firms that produce the same product have zero marginal cost of production.
- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

Cournot Competition

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- Strategy space is $S_i = [0, +\infty)$

Cournot Competition

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- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

- Strategy space is $S_i = [0, +\infty)$
- The utility function of player i is given by:

$$\pi_1(q_1, q_2) = (120 - (q_1 + q_2))q_1,$$
$$\pi_2(q_1, q_2) = (120 - (q_1 + q_2))q_2.$$

$$\pi_1(q_1, q_2) = \overbrace{(120 - q_1 - q_2)}^{P} q_1$$

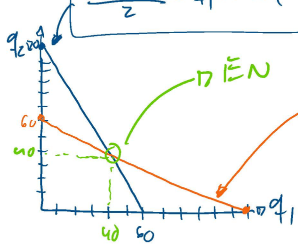
$$\frac{\partial \pi_1}{\partial q_1} = 120 - 2q_1 - q_2 = 0$$

$$\boxed{\frac{120 - q_2}{2} = q_1 = MR_1(s_2 = q_2)}$$

$$\pi_2(q_1, q_2) = (120 - q_1 - q_2) q_2$$

$$\frac{\partial \pi_2}{\partial q_2} = 120 - q_1 - 2q_2 = 0$$

$$\boxed{\frac{120 - q_1}{2} = q_2 = MR_2(s_1 = q_1)}$$



$$\frac{120 - q_2^*}{2} = q_1^* \rightarrow 120 - q_2^* - 2q_1^* = 0$$

$$\frac{120 - q_1^*}{2} = \boxed{q_2^*} \rightarrow 120 - q_1^* - 2q_2^* = 0$$

$$\times(-2)$$

$$-240 + 2q_2 + 4q_1 = 0$$

Cournot Competition

- Are there any strictly dominant strategies?

Cournot Competition

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Cournot Competition

40 60

$-240 + 2q_2 + 4q_1 = 0$

$-120 + 3q_1 = 0$

$40 = q_1^*$

$40 = q_2^*$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
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- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

- Therefore 60 strictly dominates any $q_i \in (60, 120]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$

Cournot Competition

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Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 60]$
- 45 strictly dominates all strategies $q_i \in (45, 60]$
- After three rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 45]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 45]$
- 37.5 strictly dominates all strategies $q_i \in [30, 37.5]$
- After four rounds of deletion of strictly dominated strategies, we are left with: $S_i = [37.5, 45]$

Cournot Competition

- After (infinitely) many iterations, the only remaining strategies are $S_i = 40$
- The unique solution by IDSDS is $q_1^* = q_2^* = 40$.

Cournot Competition

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- $$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$
- We can solve for q_1^* and q_2^* to obtain:

$$q_1^* = 40, q_2^* = 40, Q^* = 80, \Pi_1^* = \Pi_2^* = 1600.$$

Cournot Competition vs Monopoly (cartel)

- In a perfectly competitive market, price equals marginal cost and the total quantity produced will be $Q = 120$.

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(1) 1800 (2) 1800

$\Pi_m = (120 - Q)Q$

$\frac{\partial \Pi_m}{\partial Q} = 120 - 2Q = 0$

$\boxed{Q_m^* = 60}$

Cournot Competition vs Monopoly (cartel)

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- The profits to each firm in the Cournot Competition is less than half of the monopoly profits
- In a duopoly, externalities are imposed on the other firm

Lecture 12: Game Theory // Nash equilibrium

Dominance
  Weakly dominated strategies

Nash equilibrium

Some examples

Relationship to dominance

Examples
  Cournot Competition
  Cartels

Cartels

- Suppose there ~~are three firms who face zero~~ marginal cost
- The inverse demand function is given by:

$$p(q_1 + q_2 + q_3) = 1 - q_1 - q_2 - q_3 = 1 - Q$$

$$P(q_1, q_2, q_3) = 1 - Q$$

$$\Pi_i(q_1, q_2, q_3) = (q_i, q_{-i}) = (1 - q_i - q_{-i})q_i$$

$$\frac{\partial \Pi_i}{\partial q_i} = 1 - 2q_i - q_{-i} = 0$$

$$\boxed{\frac{1 - q_{-i}}{2} = q_i = MR_i(q_{-i})}$$

$$\frac{1 - q_1 - q_2}{2} = q_3$$

EN UN EQ. SIMETRICO

$$q_1^* = q_2^* = q_3^* = q^*$$

$$\frac{1 - q^* - q^*}{2} = q^*$$

$$1 - 2q^* = 2q^*$$

$$1 = 4q^*$$

$$\boxed{\frac{1}{4} = q^* = q_1^* = q_2^* = q_3^*}$$

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- In a Nash equilibrium we must have:

$$q_1^* = \frac{1 - q_2^* - q_3^*}{2}$$
$$q_2^* = \frac{1 - q_1^* - q_3^*}{2}$$
$$q_3^* = \frac{1 - q_1^* - q_2^*}{2}.$$

Cartels

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- $q_1^* = q_2^* = q_3^* = \frac{1}{4}$
- Price is $p^* = 1/4$ and all firms get the same profits of $1/16$

$$P = 1 - Q = 1 - \left(\frac{3}{4}\right) = \frac{1}{4}$$

$$\Pi = P \cdot q$$
$$= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

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- Therefore

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$$\Pi = (1 - q_i - q_{-i})q_i$$

$$\frac{\partial \Pi}{\partial q_i} = 1 - 2q_i - q_{-i} = 0$$

$$\frac{1 - q_{-i}}{2} = q_i$$

EN EQ SIMETRICO

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Cartels

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$\frac{1-q_{-i}}{2} = q_i$

EN EQ SIMETRICO

$q_A^* = q_B^* = q^*$

$q^* = \frac{1-q^*}{2}$

$2q^* = 1 - q^*$

$\boxed{q^* = \frac{1}{3}}$

Cartels

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- Firms 1 and 2 suffered, while firm 3 is better off!
- Firm 3 is obtaining a disproportionate share of the joint profits (more than $1/3$)

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- Firm 3 clearly wants to stay out

Cartels

There are many difficulties associated with sustaining collusive agreements (e.g., the OPEC cartel)