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Lecture13....

Lecture 13: Game Theory // Nash equilibrium

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Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued
- Cournot - Revisited
- Bertrand Competition
- Bertrand Competition - Different costs
- Bertrand Competition - 3 Firms
- Hotelling and Voting Models

Cournot Competition

- N identical firms competing on the same market

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$$p = a - b\sum_{j=1}^{N} q^j$$

Cournot Competition

- N identical firms competing on the same market
- Marginal cost is constant and equal to c
- Aggregate inverse demand is

$$p = a - b\underbrace{\boxed{\sum_{j=1}^{N} q^j}}_{Q}$$

- Benefits of firm j are:

$$\Pi^j(q^1, ...q^N) = \underbrace{\boxed{\left(a - b\sum_{i=1}^{N} q^i\right) q^j - cq^j.}}_{P}$$

$$\frac{\partial \Pi^j}{\partial q_j} = -bq^j + \left(a - b\sum_{i=1}^{N} q^i\right)(1) - c = 0$$

EN UN EQ SIMETRICO

$$\boxed{q^*_1 = q^*_2 = \dots q^*_N = q^*}$$

$$\to -bq^* + \left(a - b\underline{\sum_{i=1}^{N} q^*}\right) - c = 0$$

Cournot Competition

- The FOC for a given firm is:

$$\Pi^j(q^1, \ldots q^N) = \underbrace{\left(a - b\sum_{i=1}^{N} q^i\right)}_{P} q^j - cq^j.$$

Cournot Competition

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Cournot Competition

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- The symmetric Nash equilibrium is given by

$$q^* = \frac{a-c}{b(N+1)}$$

Cournot Competition

- The FOC for a given firm is:

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- The symmetric Nash equilibrium is given by

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- Thus

$$\begin{aligned} \sum_{j=1}^{N} q^j &= \frac{N(a-c)}{b(N+1)} \\ p &= a - N\frac{a-c}{(N+1)} < a \\ \Pi^j &= \frac{(a-c)^2}{b(N+1)^2} \end{aligned}$$

Cournot Competition

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Cournot Competition

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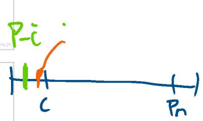
- As $N \to \infty$ we get close to perfect competition
- $N = 1$ we get the monopoly case

Lecture 13: Game Theory // Nash equilibrium

Bertrand Competition

- Consider the alternative model in which firms set prices
- In the monopolist's problem, there was not distinction between a quantity-setting model and a price setting
- In oligopolistic models, this distinction is very important



Bertrand Competition

- Consider two firms with the same marginal constant marginal cost of production and demand is completely inelastic
- Each firm simultaneously chooses a price $p_i \in [0, +\infty)$
- If p_1, p_2 are the chosen prices, then the utility functions of firm i is given by:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 0 & \text{if } p_i > p_{-i}, \\ (p_i - c)\frac{Q(p_i)}{2} & \text{if } p_i = p_{-i}, \\ (p_i - c)Q(p_i) & \text{if } p_i < p_{-i}. \end{cases} \rightarrow MR_i(P_{-i}) = \begin{cases} P_m & P_{-i} > P_m \\ P_{-i} - \varepsilon & c < P_{-i} \leq P_m \\ [c, \infty) & c = P_{-i} \\ (P_{-i}, \infty) & c > P_{-i} \end{cases}$$

Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$\begin{aligned} R(p_i) &= p_i Q(p_i) && (1) \\ MR(p_i) &= Q(p_i) + p_i Q'(p_i) && (2) \\ &= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). && (3) \end{aligned}$$

$$\rightarrow -bq^* + \underline{\left(a - b\sum_{i=1}^{N} q^*\right)} - c = 0$$

$$-bq^* + a - bNq^* - c = 0$$

$$a - c = q^* b + bNq^*$$

$$a - c = q^* b(1 + N)$$

$$\boxed{\frac{a-c}{b(N+1)} = q^*_{EN}} \qquad \boxed{Q^* = \frac{N(a-c)}{b(N+1)}}$$

$$P_{EN} = a - b\sum_{i=1}^{N} q^* = a - bNq^*$$

$$= a - \cancel{b}N\left(\frac{a-c}{\cancel{b}(N+1)}\right)$$

$$= \frac{(N+1)(a) - Na + NC}{N+1}$$

$$= \frac{\cancel{Na} + a - \cancel{aN} + NC}{N+1}$$

$$\boxed{P^* = \frac{a}{N+1} + \frac{NC}{N+1}}$$

$$\Pi^{EN} = Pq^* - Cq^* = (P - C)q^*$$

$$= \left(\frac{a + NC}{N+1} - C\right)\left(\frac{a-C}{b(N+1)}\right)$$

$$= \left(\frac{a + NC - (N+1)C}{N+1}\right)\left(\frac{a-C}{b(N+1)}\right)$$

$$= \left(\frac{a + \cancel{NC} - \cancel{NC} - C}{N+1}\right)\left(\frac{a-C}{b(N+1)}\right)$$

$$\boxed{\Pi^{EN} = \frac{(a-C)^2}{b(N+1)^2}}$$

$$\lim P^* = C$$

Bertrand Competition

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- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

Bertrand Competition

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- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.
- Then

$$MR(p_i) - c > 0 \text{ if } p_i < p^m, MR(p_i) - c < 0 \text{ if } p_i > p^m.$$

Bertrand Competition

- The best response function is:

$$BR_i(p_{-i}) = \begin{cases} p^m & \text{if } p_{-i} > p^m,\\ p_{-i} - \varepsilon & \text{if } c < p_{-i} \leq p^m,\\ [c, +\infty) & \text{if } c = p_{-i}\\ (c, +\infty) & \text{if } c > p_{-i}. \end{cases}$$

- Where ε is the smallest monetary unit

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$

Bertrand Competition

Case 1: $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$
- $BR_1(p^m - \varepsilon) = p^m - 2\varepsilon$

Bertrand Competition

Case 1: $p_1^* > p^m$

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- So this cannot be a Nash equilibrium

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$

$$\lim_{N\to\infty} P^* = c$$

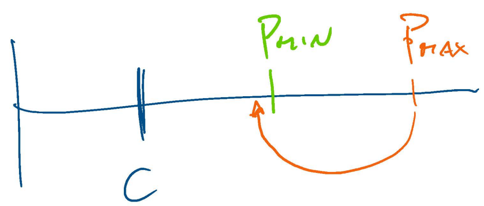
$$\lim_{N\to\infty} \Pi^* = 0$$

$$\lim_{N\to\infty} q^* = 0$$

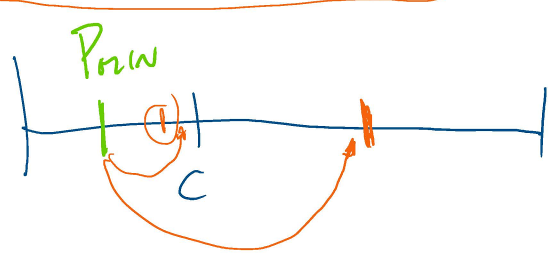
$$\lim_{N\to\infty} Q^* = \frac{a-c}{b}$$

EN EQ?

MIN$(P_1, P_2) > c$?



$\boxed{\text{MIN}(P_1, P_2) < c}$?



→ EN EQ $\boxed{\text{MIN}(P_1, P_2) = c}$

Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$

Bertrand Competition

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Bertrand Competition

Case 2: $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$
- $BR_1(p_1^* - \varepsilon) = p_1^* - 2\varepsilon$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$

Bertrand Competition

Case 3: $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$
- So this cannot be a Nash equilibrium

Bertrand Competition

Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$

Bertrand Competition

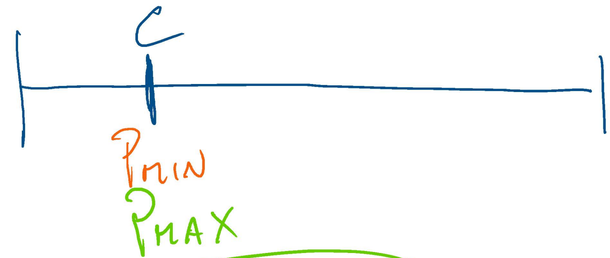
Case 4: $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$
- The unique pure strategy Nash equilibrium is $p_1^* = p_2^* = c$

Bertrand Competition

Thus in contrast to the Cournot duopoly model, in the Bertrand competition model, two firms get us back to perfect competition ($p = c$)

$\Rightarrow$ EN EQ $\boxed{MIN(P_1, P_2) = C}$



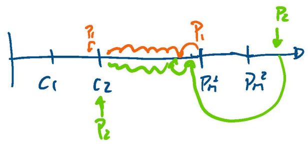
$\Rightarrow$ EN $= (P_1 = C, P_2 = C)$

Lecture 13: Game Theory // Nash equilibrium

Bertrand Competition - different costs

- Suppose that the marginal cost of firm 1 is equal to c_1 and the marginal cost of firm 2 is equal to c_2 where $c_1 < c_2$.
- The best response for each firm:

$$BR_i(p_{-i}) = \begin{cases} p_m^i & \text{if } p_{-i} > p_m^i, \\ p_{-i} - \varepsilon & \text{if } c_i < p_{-i} \leq p_m^i, \\ [c_i, +\infty) & \text{if } p_{-i} = c_i \\ (p_{-i}, +\infty) & \text{if } p_{-i} < c_i. \end{cases}$$



Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss

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Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market

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Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$, then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$, then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit

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Bertrand Competition - different costs

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- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices

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Bertrand Competition - different costs

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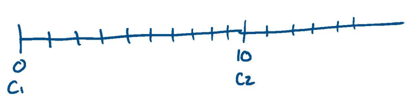
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut p_2^* and get a positive profit
- Firm 1 would really like to price at some price p_1^* just below the marginal cost of firm 2, but wherever p_2 is set, Firm 1 would try to increase prices
- No NE because of continuous prices

$$EN = \{ (P_1 = 10, P_2 = 11), (P_1 = 9, P_2 = 10), (P_1 = 8, P_2 = 9), \vdots, (P_1 = 1, P_2 = 2) \}$$

Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$



Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.

Bertrand Competition - discreet prices

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Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.
- Suppose that (p_1^*, p_2^*) is a pure strategy Nash equilibrium...

Bertrand Competition - discreet prices

Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$

Bertrand Competition - discreet prices

Case 1: $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$
- p_1^* cannot be a best response to p_2^* since by setting $p_1 = p_2^*$ firm 1 would get strictly positive profits

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price

Bertrand Competition - discreet prices

Case 2: $p_1^* \in \{1, 2, \ldots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price
- The only equilibrium is $(p_1^*, p_1^* + 1)$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:
$$\frac{1}{2}(10) = 5 < 9.$$

Bertrand Competition - discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:
$$\frac{1}{2}(10) = 5 < 9.$$
- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher

Bertrand Competition - // discreet prices

Case 3: $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:
$$\frac{1}{2}(10) = 5 < 9.$$
- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher
- $(p_1^*, p_2^*) = (10, 11)$ is a Nash equilibrium

Bertrand Competition - discreet prices

Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$

Bertrand Competition - discreet prices

Case 4: $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$
- Firm 1 would not be best responding since by setting a price of $p_1 = 10$, it would get strictly positive profits

Bertrand Competition - discreet prices

Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$

Bertrand Competition - discreet prices

Case 5: $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$
- Firm 1 is not best responding since by lowering the price it can get the whole market.

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued
- Cournot - Revisited
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Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms

EN EQ

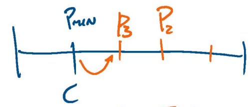
$\min(P_1, P_2, P_3) < C$ X

$\min(P_1, P_2, P_3) > C$ X

$\Rightarrow \min(P_1, P_2, P_3) = C$

P_{min} P_2 P_3

$\Rightarrow \text{MIN}(P_1, P_2, P_3) = C$



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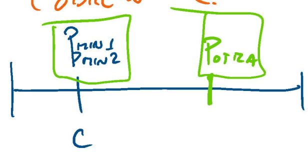

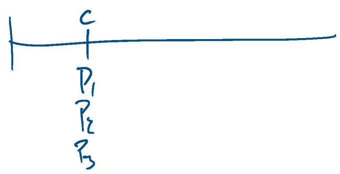

Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms
- Best response of firm i is given by:

$$BR_1(p_2, p_3) = \begin{cases} p^m & \text{if } \min\{p_2, p_3\} > p^m, \\ \min\{p_2, p_3\} - \varepsilon & \text{if } c < \min\{p_2, p_3\} \leq p^m, \\ [c, +\infty) & \text{if } c = \min\{p_2, p_3\}, \\ (\min\{p_2, p_3\}, +\infty) & \text{if } c > \min\{p_2, p_3\}. \end{cases}$$

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- (c, c, c) is indeed a pure strategy Nash equilibrium as in the two firm case

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
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Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$

Bertrand Competition - 3 firms

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- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c?

Bertrand Competition - 3 firms

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- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost

Bertrand Competition - 3 firms

- If (p_1, p_2, p_3) was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
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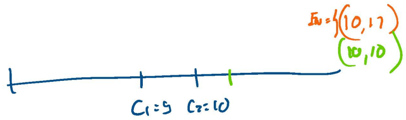
- We must have $\min\{p_1, p_2, p_3\} = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to c? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost
- Set of all pure strategy Nash equilibria are given by:

$$\{(c, c, c+\varepsilon) : \varepsilon \geq 0\} \cup \{(c, c+\varepsilon, c) : \varepsilon \geq 0\} \cup \{(c+\varepsilon, c, c) : \varepsilon \geq 0\}.$$



Lecture 13: Game Theory // Nash equilibrium

Hotelling

- Two firms $i = 1, 2$ decide to produce heterogeneous products $x_1, x_2 \in [0, 1]$

Hotelling

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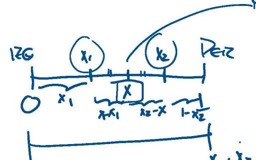

$x - x_1 = x_2 - x$

$2x = x_1 + x_2$

$x = \boxed{\dfrac{x_1 + x_2}{2}}$

$$U_1(x_1, x_2) = \begin{cases} \dfrac{x_1 + x_2}{2} & x_1 < x_2 \\ 1/2 & x_1 = x_2 \\ 1 - \left(\dfrac{x_1 + x_2}{2}\right) & x_1 > x_2 \end{cases}$$

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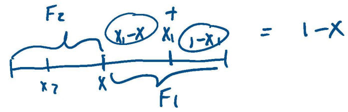

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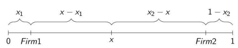

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Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

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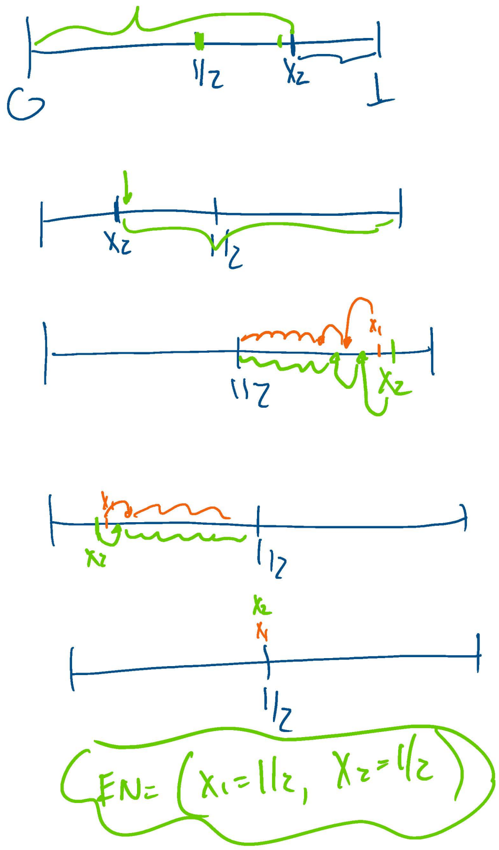

Hotelling

Compute the best response functions

- **Case 1:** Suppose first that $x_2 > 1/2$. Then setting x_1 against x_2 yields a payoff of

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- **Case 3:** Suppose next that $x_2 = 1/2$. Here there will be a best response for firm 1 at 1/2

Hotelling

$$BR_1(x_2) = \begin{cases} \emptyset & \text{if } x_2 > 1/2 \\ 1/2 & \text{if } x_2 = 1/2 \\ \emptyset & \text{if } x_2 < 1/2. \end{cases}$$

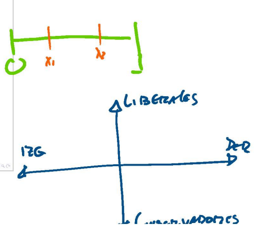
Symmetrically, we have:

$$BR_2(x_1) = \begin{cases} \emptyset & \text{if } x_1 > 1/2 \\ 1/2 & \text{if } x_1 = 1/2 \\ \emptyset & \text{if } x_1 < 1/2. \end{cases}$$

The unique Nash equilibrium is for each firm to choose $(x_1, x_2) = (1/2, 1/2)$. Each firm essentially locates in the same place

Hotelling

- Hotelling can also be done in a discreet setting
- Hotelling can be applied to a variety of situations (e.g., voting)
- But this predicts the opposite of polarization
- With three candidates, predictions are quite different
- All candidates picking $\frac{1}{2}$ is no longer a Nash equilibrium
- What are the set of pure strategy equilibria here? (this is a difficult problem).



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