Lecture 15: Game Theory // Nash equilibrium

Mauricio Romero
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Nash’s Theorem

Dynamic Games
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Nash’s Theorem

Dynamic Games
Theorem (Nash's Theorem)

Suppose that the pure strategy set $S_i$ is finite for all players $i$. A Nash equilibrium always exists.
Proof (just the intuition)

- Proof is very similar to general equilibrium proof.
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  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point

\[ X^* \text{ is a fixed point of } F(X) \iff F(X^*) = X^* \]
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- Two parts:
  1. A Nash equilibrium is a fixed point of the best response functions
  2. A finite game with mixed strategies has all the pre-requisites to guarantee a fixed point

- Remember $X^*$ is a fixed point of $F(X)$ if and only if $F(X^*) = X^*$
Proof - Part 1

- Let \((s_1^*, ..., s_n^*)\) be a Nash equilibrium.
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- Let \(\Gamma(s_1, \ldots, s_n) = (BR_1(s_{-1}), BR_2(s_{-2}), \ldots, BR_n(s_{-n}))\).

- \(\Gamma(s_1^*, \ldots, s_n^*) = (s_1^*, \ldots, s_n^*)\).

- Therefore \((s_1^*, \ldots, s_n^*)\) is a fixed point of \(\Gamma\).
Theorem (Kakutani fixed-point theorem)

Let \( \Gamma : \Omega \to \Omega \) be a correspondence that is upper semi-continuous, \( \Omega \) be non empty, compact (closed and bounded), and convex \( \Rightarrow \) \( \Gamma \) has at least one fixed point
Proof - Part 2

So we want to apply Kakutani’s theorem. If the game is finite and we allow mixed strategies then

\[
\Gamma : \Sigma \rightarrow \Sigma
\]

\(\Gamma(\sigma_1, \ldots, \sigma_n) = (BR_1(\sigma_1), BR_2(\sigma_2), \ldots, BR_n(\sigma_n))\) is upper semi-continuous. Why?

If two pure strategies are in the best response of a player \((\sigma_i, \sigma'_i \in BR_i(s_i))\), then any mixing of those strategies is also a best response (i.e., \(p\sigma + (1-p)\sigma' \in BR_i(s_i)\)).

Therefore if \(\Gamma(\sigma_1, \ldots, \sigma_n)\) has two images, those two images are connected (via all the mixed strategies that connect those two images).

That happens to be the definition of upper semi-continuous.
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- $\Sigma$ is compact: It includes the boundary (pure strategies) and is bounded (the game only has a finite set of strategies)
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Dynamic Games
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The set of Nash equilibria of the extensive form game is simply the set of all Nash equilibria of the normal form representation of the game.
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Reminder: A (pure) strategy is a complete contingent plan of action at every information set.

The set of Nash equilibria of the extensive form game is simply the set of all Nash equilibria of the normal form representation of the game.

Some of the equilibria do not make much sense intuitively.
Example 23 (Predation). Firm 1 (the entrant) can choose whether to enter a market against a single incumbent, Firm 2, or exit. If Firm 1 enters, Firm 2 can either respond by fighting or accommodating. The extensive form and payoffs are drawn in Figure 1.

To find Nash equilibria of this game, we can write out the normal form as follows.

Clearly, \((x, f)\) is a Nash equilibrium, as is \((e, a)\). However, \((x, f)\) does not seem like a plausible prediction: conditional upon Firm 1 having entered, Firm 2 is strictly better off accommodating rather than fighting. Hence, if Firm 1 enters, Firm 2 should accommodate. But then, Firm 1 should foresee this and enter, since it prefers the outcome \((e, a)\) to what it gets by playing \(x\).

The problem in the Example is that the "threat" of playing \(f\), that is fighting upon entry, is not credible. The outcome \((x, f)\) is Nash because if Firm 2 would fight upon entry, then Firm 1 is better off exiting. However, in the dynamic game, Firm 1 should not believe such an "empty threat". The crux of the matter is that the Nash equilibrium concept places no restrictions on players' behavior at nodes that are never reached on the equilibrium path. In this example, given that Firm 1 is playing \(x\), any action for Firm 2 is a best response, since all its actions are at a node that is never reached when Firm 1 places \(x\). Thus, by choosing an action \((f)\) that it certainly wouldn't want to play if it were actually forced to act, it can ensure that Firm 1's [unique, in this case] best response is to play \(x\), guaranteeing that it in fact won't have to act.

\[\text{42There are also some MSNE involving } x.\]
Two Nash equilibria: \((x,f)\) and \((e,a)\).

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In the previous example, \(f\) is not optimal if we reach the second period
A natural way to make sure players are optimizing in each node is to solve the game via backwards induction.
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**Theorem (Zermelo)**

In every finite game where every information set has a single node (i.e., complete information), has an Nash equilibrium that can be derived via backwards induction. If the payouts to players are different in all terminal nodes, then the Nash equilibrium is unique.
Theorem (Zermelo II)

In any finite two-person game of perfect information in which the players move alternatingly and in which chance does not affect the decision making process, if the game cannot end in a draw, then one of the two players must have a winning strategy (i.e. force a win).
Centipede Game
Nash equilibria are \{ (P, P), P \} and \{ (P, C), P \}.
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But by backward induction, the solution is to play \( P \) in each period
Consider the following game

```
1
A
X
Y
X
Y
3, 3
4, 2
4, 1
B
6, 7
5, 5
1
L
M
A
B
XY
2
1
3, 3 4, 2
6, 7 4, 1

Some Applications of Subgame Perfection

16-9: Mono polymer manufacturer M produces at a cost $10 per unit. M sells to R, who then sells to consumers. The inverse demand curve is \( p = 200 - \frac{q}{100} \).

The game runs as follows:

1. M chooses a price \( x \) to offer to R.
2. R observes \( x \) and then chooses how many units \( q \) to purchase.
3. M obtains profit \( u_M = q(x - 10) \); R obtains \((200 - \frac{q}{100})q - xq\).

Calculating the subgame-perfect Nash equilibrium:

Note that there are an infinite number of information sets for R, each is identified by a number \( x \), and each initiates a subgame. Calculate the equilibrium of these subgames, by finding R's optimal \( q \) as a function of \( x \)... \( q^*(x) = 10000 - 50x \).

M can anticipate this from R, so M's payoff of choosing \( x \) is \( q^*(x)(x - 10) = (10000 - 50x)(x - 10) \).

M's optimum is... \( x^* = 105 \).

Advertising and Competition

The game:

1. Firm 1 selects a nonnegative advertising level \( a \) at cost \( 2a^3/8 \).
2. Firm 2 observes \( a \) and then the two firms engage in Cournot competition, where they select quantities \( q_1 \) and \( q_2 \), produce at zero cost, and face the inverse demand curve \( p = a - q_1 - q_2 \).
```
Can’t be solved by backwards induction
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Thus, we need something else
- Can’t be solved by backwards induction

- Thus, we need something else

- First, we need to defined a subgame
A sub-game, of a game in extensive form, is a sub-tree such that

- It starts in a single node
- If contains a node, it contains all subsequent nodes
- If it contains a node in an information set, it contains all nodes in the information set
Definition
A subgame of an extensive form game is the set of all actions and nodes that follow a particular node that is not included in an information set with another distinct node.
By definition, the original game is a subgame
Example 23 (Predation). Firm 1 (the entrant) can choose whether to enter a market against a single incumbent, Firm 2, or exit. If Firm 1 enters, Firm 2 can either respond by fighting or accommodating. The extensive form and payoffs are drawn in Figure 1.

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There are also some MSNE involving \(x\).
Centipede Game

```
(1,0)  C  (0,2)  C  (4,1)  C  (3,3)
     P    P    P
```

1 2 1
P P P
C C C
(1,0) (0,2) (4,1)

---

The Centipede Game is a sequential game where two players move sequentially. The game starts with a payoff of (1,0) and each subsequent move reduces the payoff by (0,1). The game ends when a player chooses to stop, and the other player receives the remaining payoff. The optimal strategy for both players is to continue moving at each stage until the last move, as the payoff would be zero otherwise.
Some Applications of Subgame Perfection

16-9: M produces at a cost $10 per unit. M sells to R, who then sells to consumers. The inverse demand curve is $p = 200 - q/100$.

The game runs as follows:

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2. Firm 2 observes $a$ and then the two firms engage in Cournot competition, where they select quantities $q_1$ and $q_2$, produce at zero cost, and face the inverse demand curve $p = a - q_1 - q_2$.
Since in some games (where multiple nodes are in the same information set) we can’t formally choose how people are optimizing, we extend the notion of backwards induction to subgames

**Definition (Subgame perfect Nash equilibria)**

A pure strategy profile is a Subgame perfect Nash equilibria (SPNE) if and only if it involves the play of a NE in every subgame of the game.
Remark

Every SPNE is a NE

Remark

As in normal form games, mixed strategy SPNE can be defined but this is a bit technical. Thus, we will not worry about it for the purposes of the course.
Some Applications of Subgame Perfection

16-9: MANFACTURER/RETAILER
M produces at a cost $10 per unit. M sells to R, who then sells to consumers. The inverse demand curve is $p = 200 - q/100$.

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11

Advertising and Competition

The game:
1. Firm 1 selects a nonnegative advertising level $a$ at cost $3a/8$.
2. Firm 2 observes $a$ and then the two firms engage in Cournot competition, where they select quantities $q_1$ and $q_2$, produce at zero cost, and face the inverse demand curve $p = a - q_1 - q_2$.

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Some Applications of Subgame Perfectness

10 Calculating the subgame - perfect Nash equilibrium:

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q*(x) = 10000 – 500x.

M can anticipate this from R, so M's payoff of choosing x is q*(x)(x – 100) = (10000 – 500x)(x – 100).

M's optimum is... x* = 105.

11 Advertising and Competition

The game:

1. Firm 1 selects a nonnegative advertising level a at cost 2a / 8.

2. Firm 2 observes a and then the two firms engage in Cournot competition, where they select quantities q1 and q2, produce at zero cost, and face the inverse demand curve p = a – q1 – q2.

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The game has 3 NE: (LB,X), (MA,Y), (MB,Y)

The subgame has a single NE: (B,X)

The SPNE is (LB,X)