Lecture16

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Lecture 16: Applications of Subgame Perfect Nash Equilibrium

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Ultimatum Game

Alternating offers

Stackelberg Competition

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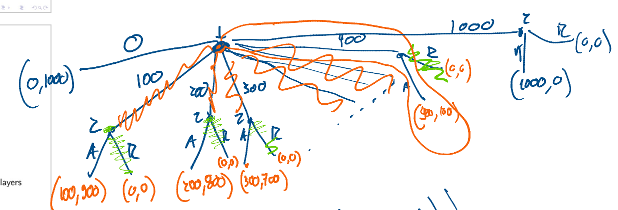
Alternating offers

Stackelberg Competition

1. Player 1 makes a proposal $(x, 1000 - x)$ of how to split 1000 pesos among $(100, 900), \ldots, (800, 200), (900, 100)$
2. Player 2 accepts or rejects the proposal
3. If player 2 rejects both obtain 0. If 2 accepts, then the payoffs or the two players are determined by $(x, 1000 - x)$



$$EPS : \left(S_1 : 900,\ S_2 = (\underbrace{AAA\ldots A}_{9\text{ veces}}) \right)$$

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- In any SPNE, player 1 makes the proposal $(900, 100)$

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- When extreme offers like $(900, 100)$ are made, player 2 rejects in many cases
- Player 2 may care about inequality or positive utility associated with "punishment" aversion

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- Two players are deciding how to split a pie of size 1

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- The players would rather get an agreement today than tomorrow (i.e., discount factor)

- Player 1 makes an offer θ_1

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- If player 1 accepts or rejects the proposal
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- Player 1 makes an offer θ_1
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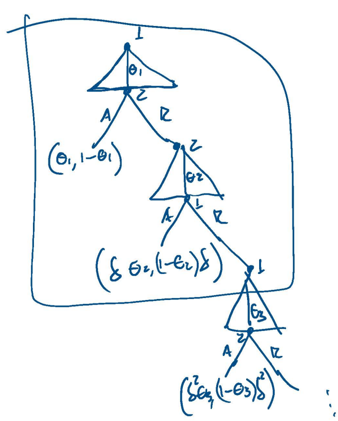
- If player 1 accepts or rejects the proposal
- If player 1 rejects, player 1 makes an offer θ_3
- ... and on and on for T periods

- Player 1 makes an offer $\boxed{\theta_1}$
- Player 2 accepts or rejects the proposal
- If player 2 rejects, player 2 makes an offer θ_2
- If player 1 accepts or rejects the proposal
- If player 1 rejects, player 1 makes an offer θ_3
- ... and on and on for T periods
- If no offer is ever accepted, both payoffs equal zero



$\boxed{T=1}$

S_2 ACEPTA

SI $1-\theta_1 \geq 0$

$\theta_1 \leq 1$

S_1

$\boxed{\theta_1 = 1}$

PAGOS $(1,0)$

The discount factor is $\delta \leq 1$.
If Player 1 offer is accepted by Player 2 in round m,

$$\pi_1 = \delta^m \theta_m,$$

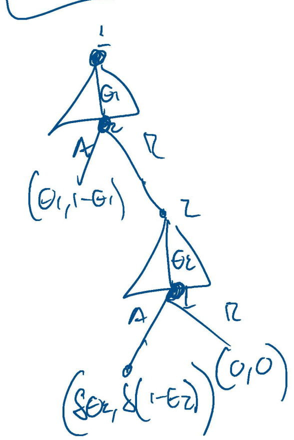
$$\pi_2 = \delta^m (1-\theta_m).$$

If Player 2 offer is accepted, reverse the subscripts

- Consider first the game without discounting

- Consider first the game without discounting
- There is a unique SPNE:

$\boxed{T=2}$



- Consider first the game without discounting
- There is a unique SPNE:

- Consider first the game without discounting
- There is a unique SPNE: The player that makes the last offer gets the whole pie
- Last-mover advantage

- In the game with discounting, the total value of the pie is 1 in the first period, δ in the second, and so forth

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- In period $(T-1)$, Player 2 could offer Smith δ, keeping $(1-\delta)$ for himself

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- Player 2 would accept (indifferent between accepting and rejecting)
- In period $(T-1)$, Player 2 could offer Smith δ, keeping $(1-\delta)$ for himself
- Player 1 would accept (indifferent between accepting and rejecting) since the **whole pie** in the next period is worth δ

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself

$\boxed{J_1}$ A $\delta\theta_2 \geq 0$

$\theta_2 \geq 0$

$\boxed{J_2}$ $\theta_2 = 0$

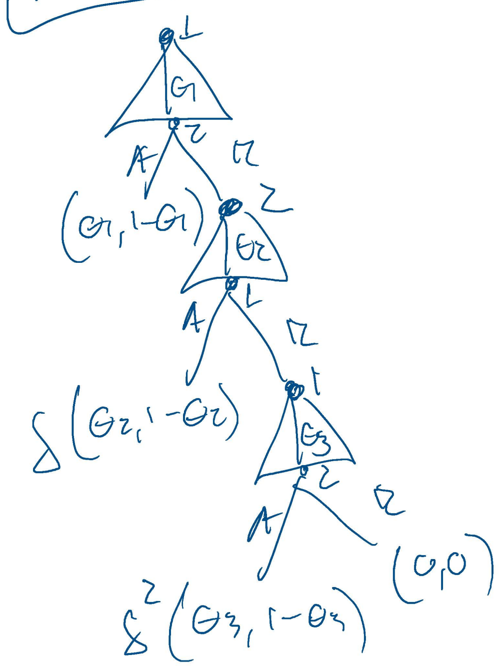
PAGOS $(0, \delta)$

$\boxed{J_2}$ A si $1-\theta_1 \geq \delta$

$\boxed{1-\delta \geq \theta_1}$

$\boxed{J_1}$ $\boxed{\theta_1 = 1-\delta} \to \boxed{\text{PAGOS } (1-\delta, \delta)}$

$\boxed{T=3}$



$\boxed{J_2}$ A si $1-\theta_3 \geq 0$

$\theta_3 \leq 1$

$\boxed{J_1}$ $\theta_3 = 1 \to$ PAGOS $(\delta^2, 0)$

$\boxed{J_1}$ A si $\delta\theta_2 \geq \delta^2$

$\theta_2 \geq \delta$

$\boxed{J_2}$ $\theta_2 = \delta$

PAGOS $(\delta \cdot \delta, \delta(1-\delta))$

$\boxed{J_2}$ ACEPTA

whole pie in the next period is worth δ

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself

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- In period $(T-3)$, Player 2 would offer Player 1 $\delta[1-\delta(1-\delta)]$, keeping $(1-\delta[1-\delta(1-\delta)])$ for himself

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- Player 2 would accept since he can earn $(1-\delta)$ in the next period, which is worth $\delta(1-\delta)$ today
- In period $(T-3)$, Player 2 would offer Player 1 $\delta[1-\delta(1-\delta)]$, keeping $(1-\delta[1-\delta(1-\delta)])$ for himself

- Player 1 would accept...

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- Player 1 would accept...
- ...

- In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting

Table 1 shows the progression of Player 1's shares when $\delta = 0.9$.

Table 1: Alternating Offers over Finite Time

Round	**1's share**	**2's share**	**Total value**	**Who offers?**
$T-3$	$\delta(1-\delta(1-\delta))$	$1-\delta(1-\delta(1-\delta))$	δ^{T-4}	2
$T-2$	$1-\delta(1-\delta)$	$\delta(1-\delta)$	δ^{T-3}	1
$T-1$	δ	$1-\delta$	δ^{T-2}	2
T	1	0	δ^{T-1}	1

- If $T = 3$ (i.e, 1 offers, 2 offers, 1 offers)

$\boxed{S_2}$ Acepta

$$1-\theta_1 \geq \delta(1-\delta)$$

$$1-\delta(1-\delta) \geq \theta_1$$

$\boxed{S_1}$ $\boxed{\theta_1 = 1-\delta(1-\delta)}$

PAGOS $\underline{\left(1-\delta(1-\delta),\ \delta(1-\delta)\right)}$

$$1-0.9(0.1),\quad 0.9(0.1)$$

- If $T = 3$ (i.e, 1 offers, 2 offers, 1 offers)
- One offers $\delta(1-\delta)$, 2 accepts in period 1

- Player 1 always does a little better when he makes the offer than when Player 2 does

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- Player 1 always does a little better when he makes the offer than when Player 2 does
- If we consider just the class of periods in which Player 1 makes the offer, Player 1's share falls

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$T=1 \quad (1,0)$

$T=2 \quad (\delta, 1-\delta)$

$T=3 \quad (\underline{1-\delta(1-\delta)}, \delta(1-\delta))$

$T=4$

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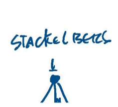
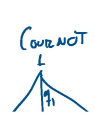
- Suppose that the inverse demand function is given by:

$$P(q_1 + q_2).$$





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- Suppose instead that the firms move in sequence which is called a **Stackelberg competition game**
- Suppose that the inverse demand function is given by
  $$\boxed{P(q_1 + q_2)}$$
- Firms have the cost functions $\boxed{c_i(q_i)}$.

The timing of the game is given by:

1. First firm 1 chooses $q_1 \geq 0$
2. Second firm 2 observes the chosen q_1 and then chooses q_2

- The game tree in this game is then depicted by an infinite tree

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- A pure strategy for firm 1 is just a choice of $q_1 \geq 0$
- A strategy for firm 2 specifies what it does after every choice of q_1
- Firm 2's strategy is a function $q_2(q_1)$ which specifies exactly what firm 2 does if q_1 is the chosen strategy of player 1

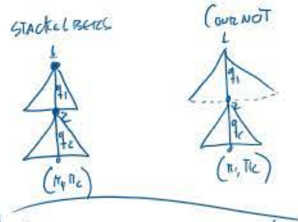
The utility functions for firm i when firm 1 chooses q_1 and firm 2 chooses the strategy (or function) $q_2(\cdot)$ is given by:

$$u_1(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_1 - c_1(q_1)$$
$$u_2(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_2(q_1) - c_2(q_2(q_1))$$

- There are many Nash equilibria of this game which are a bit counter intuitive

STACKELBERG

COURNOT



$$S_1 = \{q_1 : q_1 \in \mathbb{R}^+\}$$

$$S_2 = \{f : \underset{q_1}{\mathbb{R}^+} \longrightarrow \underset{q_2}{\mathbb{R}^+}\}$$

$$\Pi_1 = P(q_1 + q_2(q_1))q_1 - C_1(q_1)$$

$$\Pi_2 = P(q_1 + q_2(q_1))q_2(q_1) - C_2(q_2(q_1))$$

EPS

F_2

$$\Pi_2 = \overbrace{(A - q_1 - q_2)}^{P} q_2 - \cancel{C(q_2)} = 0$$

$$\frac{\partial \Pi_2}{\partial q_2} = 0 = A - q_1 - 2q_2 = 0$$

$$q_2 = \frac{A - q_1}{2}$$

$$\boxed{q_2(q_1) = \frac{A - q_1}{2}}$$

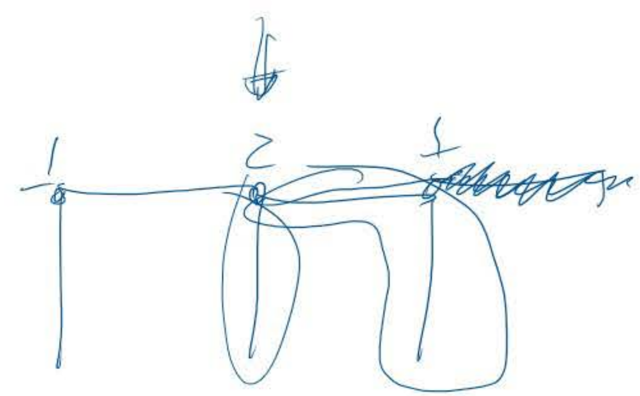
$\boxed{T=1}$ F_1

$$\Pi_1 = (A - q_1 - q_2) q_1$$

$$= \left(A - q_1 - \left(\frac{A - q_1}{2}\right)\right) q_1$$

$$= \left(\frac{A - q_1}{2}\right) q_1$$

$$\frac{\partial \Pi_1}{} = \frac{A - 2q_1}{} = 0$$



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- Cconsider the following specific game with demand function given by:

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- Cconsider the following specific game with demand function given by:

$$P(q_1 + q_2) = A - q_1 - q_2.$$

- Let the marginal costs of both firms be zero
- Then the normal form simplifies:

$$\Pi_1 = u_1(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_1,$$
$$\Pi_2 = u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

- What is an example of a Nash equilibrium of this game?

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- Let $\alpha \in [0, A)$ and consider the following strategy profile:

$$q_1^* = \alpha, q_2^*(q_1) = \begin{cases} A & \text{if } q_1 \neq \alpha, \\ \frac{A-\alpha}{2} & \text{if } q_1 = \alpha. \end{cases}$$

- What is an example of a Nash equilibrium of this game?
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- Let us check that indeed this constitutes a Nash equilibrium

- First we check the best response of player 1

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- If player 2 plays q_2^*, then player 1's utility function is given by:

$$u_1(q_1, q_2^*(\cdot)) = \begin{cases} (A - \alpha - (\frac{A-\alpha}{2}))\,\alpha > 0 & \text{if } q_1 = \alpha \\ -q_1^2 \leq 0 & \text{if } q_1 \neq \alpha. \end{cases}$$

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{A - 2q_1}{2} = 0$$

$$\boxed{\frac{A}{2} = q_1^*}$$

$$\boxed{\Pi_1^*} = (A - q_1 - q_2)\,q_1 = \left(A - \frac{A}{2} - q_2\right)\frac{A}{2}$$

$$q_2 = \frac{A - q_1}{2}$$

$$= \left(A - \frac{A}{2} - \left(\frac{A - \frac{A}{2}}{2}\right)\right)\frac{A}{2}$$

$$= \left(\frac{A - \frac{A}{2}}{2}\right)\frac{A}{2}$$

$$= \frac{A^2}{8}$$

$$\Pi_2^* = (A - q_1 - q_2)(q_2)$$

$$= \left(A - \frac{A}{2} - \left(\frac{A - \frac{A}{2}}{2}\right)\right)\left(\frac{A - \frac{A}{2}}{2}\right)$$

$$\Rightarrow \Pi_1\left(s_2 = \frac{A}{\ldots}\right) = (A - q_1 - q_2)\,q_1$$

- First we check the best response of player
- If player 2 plays q_2 then player 1's utility function is given by:

$$u_1(q_1, q_2(\cdot)) = \begin{cases} (A - c - (\frac{A-c}{2}))c > 0 & \text{if } q_1 = c \\ -q_1^2 < 0 & \text{if } q_1 \neq c \end{cases}$$

- Thus:

$$\max_{q_1 \geq 0} u_1(q_1, q_2(\cdot))$$

is solved at $q_1^* = c$.

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- Firm 1 is best responding to player 2's strategy.

- Suppose that firm 1 plays the strategy q_1^*. Is firm 2 best responding?
- Firm 2's utility function is given by:

$$u_2(q_1^*, q_2(\cdot)) = (A - c - q_2(c))q_2(c).$$

- Thus, firm 2 wants to choose the optimal strategy $q_2(\cdot)$ that maximizes the following utility

$$\max_{q_2(\cdot)} (A - c - q_2(c))q_2(c)$$

- By the first order condition, we know that

$$q_2(c) = \frac{A-c}{2}$$

- The utility function of firm 2 does not depend at all on what it chooses for $q_2(q_1)$ when $q_1 \neq c$.

$$= \left(\frac{A-\frac{A}{2}}{2}\right)\left(\frac{A-\frac{A}{2}}{2}\right) = \frac{A^2}{16}$$

EPS: $\cancel{\left(s_1 = \frac{A}{2},\ s_2 = \frac{A-\frac{A}{2}}{2} = \frac{A}{4}\right)}$

$$\frac{A-q_1}{2}$$

$$\left(s_1 = \frac{A}{2},\ s_2 = \frac{A-q_1}{2}\right)$$

$$\rightarrow \pi_1\left(s_2 = \frac{A}{4}\right) = (A - q_1 - q_2)q_1$$

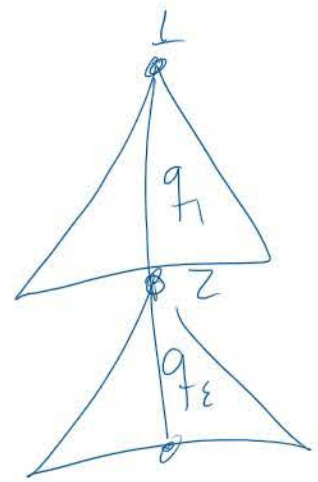
$$= \left(A - q_1 - \frac{A}{4}\right)q_1$$

$$\frac{\partial \pi_1}{\partial q_1} = A - 2q_1 - \frac{A}{4} = 0$$

$$\frac{3A}{4} = 2q_1$$

$$\frac{3A}{8} = q_1$$

EN



$$s_2 = \begin{cases} q_2 = A & \text{si } q_1 > 0 \\ q_2 = q^M & \text{si } q_1 = 0 \end{cases}$$

Firma 1: $\pi_1(q_1, q_2) = \begin{cases} \cancel{(A - q_1 - A)}q_1 \to 0 & q_1 > 0 \\ 0 & q_1 = 0 \end{cases}$

- Thus, firm 2 wants to choose the optimal strategy $q_2(\cdot)$ that maximizes the following utility:

$$\max_{q_2(\cdot)}(A - \alpha - q_2(\alpha))q_2(\alpha)$$

- By the first order condition, we know that

$$q_2(\alpha) = \frac{A - \alpha}{2}.$$

- The utility function of firm 2 does not depend at all on what it chooses for $q_2^*(q_1)$ when $q_1 \neq \alpha$
- In particular, q_2^* is a best response for firm 2

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- The equilibrium outcome of the above Nash equilibrium above is that firm 1 sets the price α and firm 2 sets the price $(A - \alpha)/2$.

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- The equilibrium outcome of the above Nash equilibrium above is that firm 1 sets the price α and firm 2 sets the price $(A - \alpha)/2$.
- In particular, in the Nash equilibrium corresponding to $\alpha = 0$, the equilibrium outcome is for firm 1 to choose a quantity of 0 and firm 2 setting a price of $A/2$
- This would be the same outcome if firm 2 were the monopolist in this market

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- Consider the equilibrium in which $\alpha = 0$

$\Rightarrow MR_1(S_2) = \{ q_1 \in [0, \infty) \}$

FIRMA 2 $\quad MR_2(q_1 = 0) = q^M$

$EN = (q_1 = 0, S_2)$

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- Consider the equilibrium in which $\alpha = 0$
- This equilibrium is highly counterintuitive because firm 2 obtains monopoly profits
- The reason is that essentially firm 2 is playing a strategy that involves **non-credible threats**
- Firm 2 is threatening to overproduce if firm 1 produces anything at all
- As a result, the best that firm 1 can do is to produce nothing
- If firm 1 were to hypothetically choose $q_1 > 0$, then firm 2 would obtain negative profits if it indeed follows through with $q_2^*(q_1)$.

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- Many Nash equilibria are counterintuitive in the Stackelberg game
- To eliminate such counterintuitive equilibria, we focus instead on SPNE instead of NE
- Lets continue with the setting in which marginal costs are zero and the demand function is given by $A - q_1 - q_2$

- We always start with the smallest/last subgames which correspond to the decisions of firm 2 after firm 1's choice of q_1 has been made

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- The utility function of firm 2 is given by:

$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

- We always start with the smallest/last subgames which correspond to the decisions of firm 2 after firm 1's choice of q_1 has been made
- The utility function of firm 2 is given by:

$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

- So, player 2 solves:

$$\max_{q_2(\cdot)} (A - q_1 - q_2(q_1))q_2(q_1).$$

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- **Case 2:** $q_1 \leq A$
- In this case, the first order condition implies:

$$q_2^*(q_1) = \frac{A - q_1}{2}.$$

- Thus, in any SPNE, player 2 must play the following strategy:

$$q_2^*(q_1) = \begin{cases} \frac{A-q_1}{2} & \text{if } q_1 \leq A \\ 0 & \text{if } q_1 > A. \end{cases}$$

- Then player 1's utility function given that player 2 plays q_2^* is given by:

$$u_1(q_1, q_2^*(\cdot)) = q_1(A - q_1 - q_2^*(q_1)) = \begin{cases} q_1(A - q_1) & \text{if } q_1 > A, \\ q_1\frac{A-q_1}{2} & \text{if } q_1 \leq A. \end{cases}$$

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- Thus, firm 1 maximizes:

$$\max_{q_1 \in [0,A]} q_1 \frac{A - q_1}{2}.$$

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- The **equilibrium outcome** is for firm 1 to choose $A/2$ and firm 2 to choose $A/4$

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- Lets solve for the set of SPNE (which is the same as NE) in the Cournot game with the same demand function and same costs
- In this case, (q_1^*, q_2^*) is a NE if and only if

$$q_1^* \in BR_1(q_2^*), q_2^* \in BR_2(q_1^*).$$

- For $q_1^* \in BR_1(q_2^*)$, we need q_1^* to solve the following maximization problem:

$$\max_{q_1 \geq 0}(A - q_1 - q_2^*)q_1.$$

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- As a result, solving these two equations, we get:

$$q_1^* = q_2^* = \frac{A}{3}.$$

In the Cournot game, note that firms' payoffs are:

$$\pi_1^c = \frac{A^2}{9}, \pi_2^c = \frac{A^2}{9}.$$

As we already saw, this was not Pareto efficient since each firm is getting a payoff that is strictly less than $1/2$ of the monopoly profits.

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- Thus, the firms' payoffs in the SPNE is:

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- But by choosing something optimal, firm 1 will be able to do even better