Lecture16

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Lecture 16: Applications of Subgame Perfect Nash Equilibrium

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Lecture 16: Applications of Subgame Perfect Nash Equilibrium

- Ultimatum Game
- Alternating offers
- Stackelberg Competition

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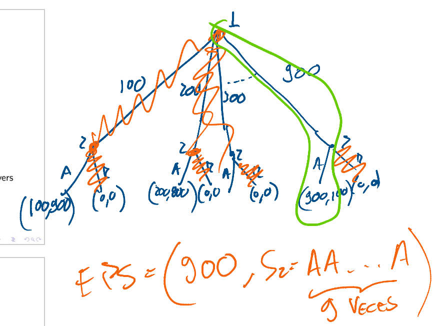
1. Player 1 makes a proposal $(x, 1000 - x)$ of how to split 1000 pesos among $(100, 900), \ldots, (800, 200), (900, 100)$
2. Player 2 accepts or rejects the proposal
3. If player 2 rejects both obtain 0. If 2 accepts, then the payoffs or the two players are determined by $(x, 1000 - x)$



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- In any SPNE, player 1 makes the proposal $(900, 100)$

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- When extreme offers like (900, 100) are made, player 2 rejects in many cases
- Player 2 may care about inequality or positive utility associated with "punishment" aversion

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Ultimatum Game

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Ultimatum Game

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Stackelberg Competition

- Two players are deciding how to split a pie of size 1

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- The players would rather get an agreement today than tomorrow (i.e., discount factor)

- Player 1 makes an offer θ_1

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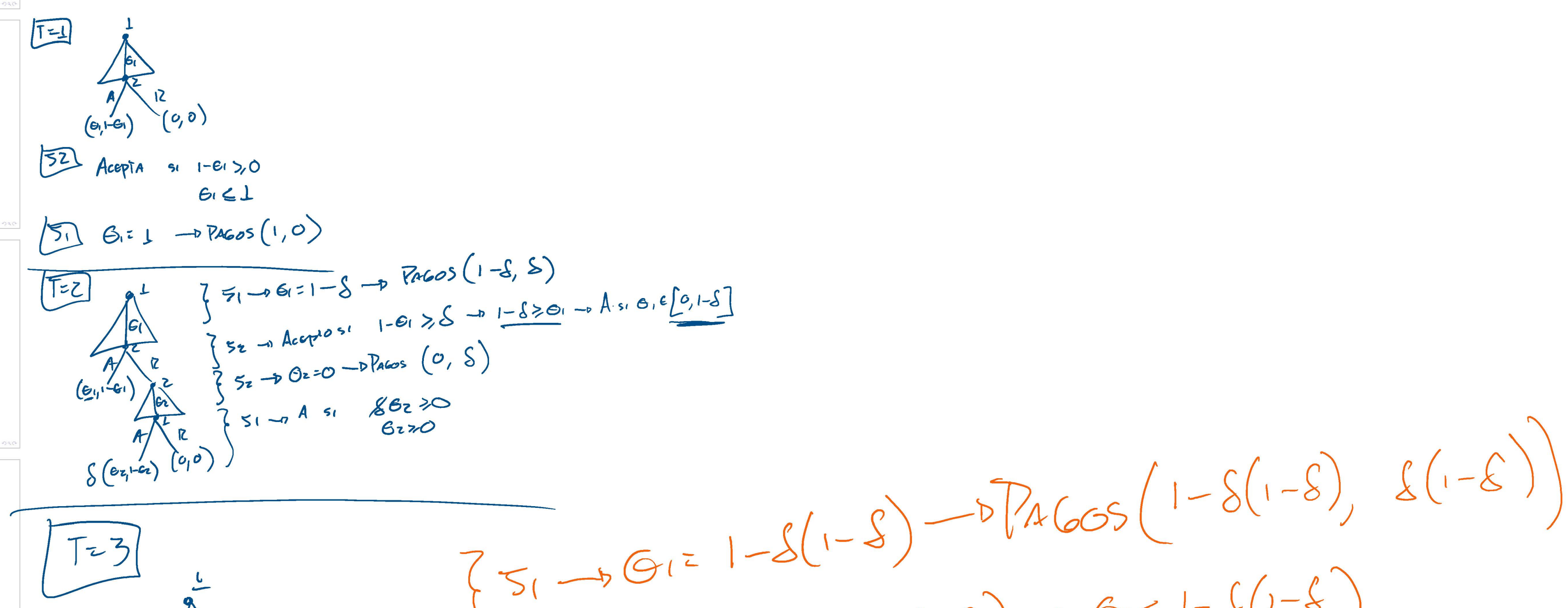
- ... and on and on for T periods

- Player 1 makes an offer θ_1
- Player 2 accepts or rejects the proposal
- If player 2 rejects, player 2 makes an offer θ_2
- If player 1 accepts or rejects the proposal
- If player 1 rejects, player 1 makes an offer θ_3
- ... and on and on for T periods
- If no offer is ever accepted, both payoffs equal zero



The discount factor is $\delta \leq 1$.
If Player 1 offer is accepted by Player 2 in round m,

$$\pi_1 = \delta^m \theta_m,$$
$$\pi_2 = \delta^m (1-\theta_m).$$

If Player 2 offer is accepted, reverse the subscripts

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- There is a unique SPNE: The player that makes the last offer gets the whole pie
- Last-mover advantage

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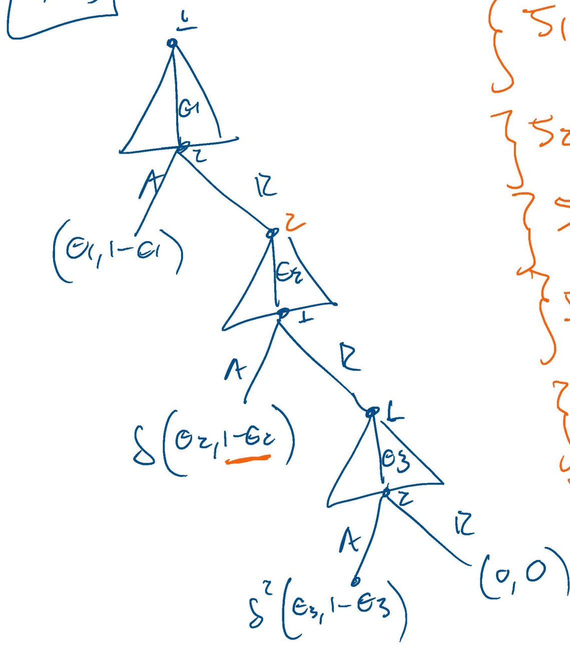
- Player 2 would accept (indifferent between accepting and rejecting)

$T=3$



$S_1 \to \theta_1 = 1-\delta(1-\delta)$

$S_2 \to A$ si $1-\theta_1 \geq \delta(1-\delta) \to \theta_1 \leq 1-\delta(1-\delta)$

$S_2 \to \theta_2 = \delta \to$ PAGOS $(\delta\cdot\delta, \delta(1-\delta))$

S_1 A si $\delta\theta_2 \geq \delta^2 \to \theta_2 \geq \delta$

S_1 $\theta_3 = 1 \to$ PAGOS $(\delta^2, 0)$

S_2 A si $\delta^2(1-\theta_3) \geq 0$

$\theta_3 \leq 1$

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- Assume Player 1 makes the last offer
- In period T, if it is reached, Player 1 would offer 0 to Player 2
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- In period $(T-1)$, Player 2 could offer Smith δ, keeping $(1-\delta)$ for himself

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- Player 2 would accept (indifferent between accepting and rejecting)
- In period $(T-1)$, Player 2 could offer Smith δ, keeping $(1-\delta)$ for himself
- Player 1 would accept (indifferent between accepting and rejecting) since the **whole pie** in the next period is worth δ

- In period $(T-2)$, Player 1 would offer Player 2 $\delta(1-\delta)$, keeping $(1-\delta(1-\delta))$ for himself

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- Player 1 would accept...
- ...
- In equilibrium, the very first offer would be accepted, since it is chosen precisely so that the other player can do no better by waiting

Table 1 shows the progression of Player 1's shares when $\delta = 0.9$.

Table 1: Alternating Offers over Finite Time

Round	1's share	2's share	Total value	Who offers?
$T-3$	$\delta(1-\delta(1-\delta))$	$1-\delta(1-\delta(1-\delta))$	δ^{T-4}	2
$T-2$	$1-\delta(1-\delta)$	$\delta(1-\delta)$	δ^{T-3}	1
$T-1$	δ	$1-\delta$	δ^{T-2}	2
T	1	0	δ^{T-1}	1

- If $T = 3$ (i.e, 1 offers, 2 offers, 1 offers)

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- One offers $\delta(1-\delta)$, 2 accepts in period 1

- Player 1 always does a little better when he makes the offer than when Player 2 does

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- If we consider just the class of periods in which Player 1 makes the offer, Player 1's share falls

Ultimatum Game

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- Suppose instead that the firms move in sequence which is called a **Stackelberg competition game**
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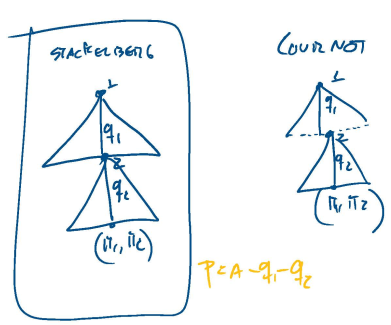
- Firms have the cost functions $c_i(q_i)$.



$P = A - q_1 - q_2$

Te timing of the game is given by:

1. First Firm 1 chooses $q_1 \geq 0$
2. Second Firm 2 observes the chosen q_1 and then chooses q_2

- The game tree in this game is then depicted by an infinite tree

- Let us write down the normal form representation of this game.

$$S_1 = \{ q_1 \in \mathbb{R}^+ \},$$

$$S_2 = \{ f : \mathbb{R}^+ \to \mathbb{R}^+ \} =$$

($\mathbb{R}^+ \ni q_1$, $\mathbb{R}^+ \ni q_2$)

$$\Pi_1(q_1, q_2(q_1)) = P(q_1 + q_2(q_1))q_1 - C_1(q_1)$$

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- A pure strategy for firm 1 is just a choice of $q_1 \geq 0$
- A strategy for firm 2 specifies what it does after every choice of q_1
- Firm 2's strategy is a function $q_2(q_1)$ which specifies exactly what firm 2 does if q_1 is the chosen strategy of player 1

The utility functions for firm i when firm 1 chooses q_1 and firm 2 chooses the strategy (or function) $q_2(\cdot)$ is given by:

$$\pi_1(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_1 - c_1(q_1)$$
$$\pi_2(q_1, q_2(\cdot)) = P(q_1 + q_2(q_1))q_2(q_1) - c_2(q_2(q_1))$$

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- Cconsider the following specific game with demand function given by:

$$P(q_1 + q_2) = A - q_1 - q_2.$$

- Let the marginal costs of both firms be zero
- Then the normal form simplifies:

$$u_1(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_1,$$
$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

$$\Pi_1(q_1, q_2(q_1)) = P(q_1 + q_2(q_1))q_1 - C_1(q_1)$$
$$\Pi_2(q_1, q_2(q_1)) = P(q_1 + q_2(q_1))q_2(q_1) - C_2(q_2(q_1))$$

$$C_1(q_1) = C_2(q_2) = 0$$

$$P(q_1 + q_2) = A - q_1 - q_2$$

EPS :

$$\Pi_2(q_1, q_2) = (A - q_1 - q_2)\, q_2$$

$$\frac{\partial \Pi_2}{\partial q_2} = A - q_1 - 2q_2 = 0$$

$$\frac{A - q_1}{2} = q_2(q_1)$$

$$\Pi_1(q_1, q_2) = (A - q_1 - q_2)\, q_1$$
$$= \left(A - q_1 - \left(\frac{A - q_1}{2}\right)\right) q_1$$
$$= \left(\frac{A - q_1}{2}\right) q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{A - 2q_1}{2} = 0$$

$$\frac{A}{2} = q_1^*$$

$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

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- Let $\alpha \in [0, A)$ and consider the following strategy profile:

$$q_1^* = \alpha, q_2^*(q_1) = \begin{cases} A & \text{if } q_1 \neq \alpha, \\ \frac{A-\alpha}{2} & \text{if } q_1 = \alpha. \end{cases}$$

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- Let us check that indeed this constitutes a Nash equilibrium

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- Firm 1 is best responding to player 2's strategy.

- Suppose that firm 1 plays the strategy q_1^*. Is firm 2 best responding?

$$\frac{A}{2} = q_1$$

$$\Pi_1^* = (A - q_1 - q_2)q_1 = \left(A - q_1 - \left(\frac{A-q_1}{2}\right)\right)q_1 = \left(\frac{A-q_1}{2}\right)q_1$$

$$= \left(\frac{A - \frac{A}{2}}{2}\right)\left(\frac{A}{2}\right) = \frac{A^2}{8}$$

$$\Pi_2^* = (A - q_1 - q_2)q_2 = \left(A - q_1 - \left(\frac{A-q_1}{2}\right)\right)\left(\frac{A-q_1}{2}\right)$$

$$= \left(\frac{A-q_1}{2}\right)\left(\frac{A-q_1}{2}\right) = \left(\frac{A-q_1}{2}\right)^2 = \left(\frac{A-\frac{A}{2}}{2}\right)^2 = \frac{A^2}{16}$$

$$\text{EPS} = \left(S_1 = \frac{A}{2},\ S_2 = \frac{A - \frac{A}{2}}{2} = \frac{A}{4}\right)$$

$$\left(S_1 = \frac{A}{2},\ S_2 = \frac{A - q_1}{2}\right) \checkmark$$

$$\Pi_1\left(q_1, \frac{A}{4} = q_2\right) = \left(A - q_1 - \frac{A}{4}\right)q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = A - 2q_1 - \frac{A}{4} = 0$$

$$\frac{3A}{4} = 2q_1$$

$$\frac{3A}{8} = q_1$$

¿EN Q' NO SEA EPS?

$$S_2 = \begin{cases} q_2 = A & \text{si } q_1 > 0 \\ q_2 = q^m & q_1 = 0 \end{cases}$$

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- Thus, firm 2 wants to choose the optimal strategy $q_2(\cdot)$ that maximizes the following utility:

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$$q_2(\alpha) = \frac{A - \alpha}{2}.$$

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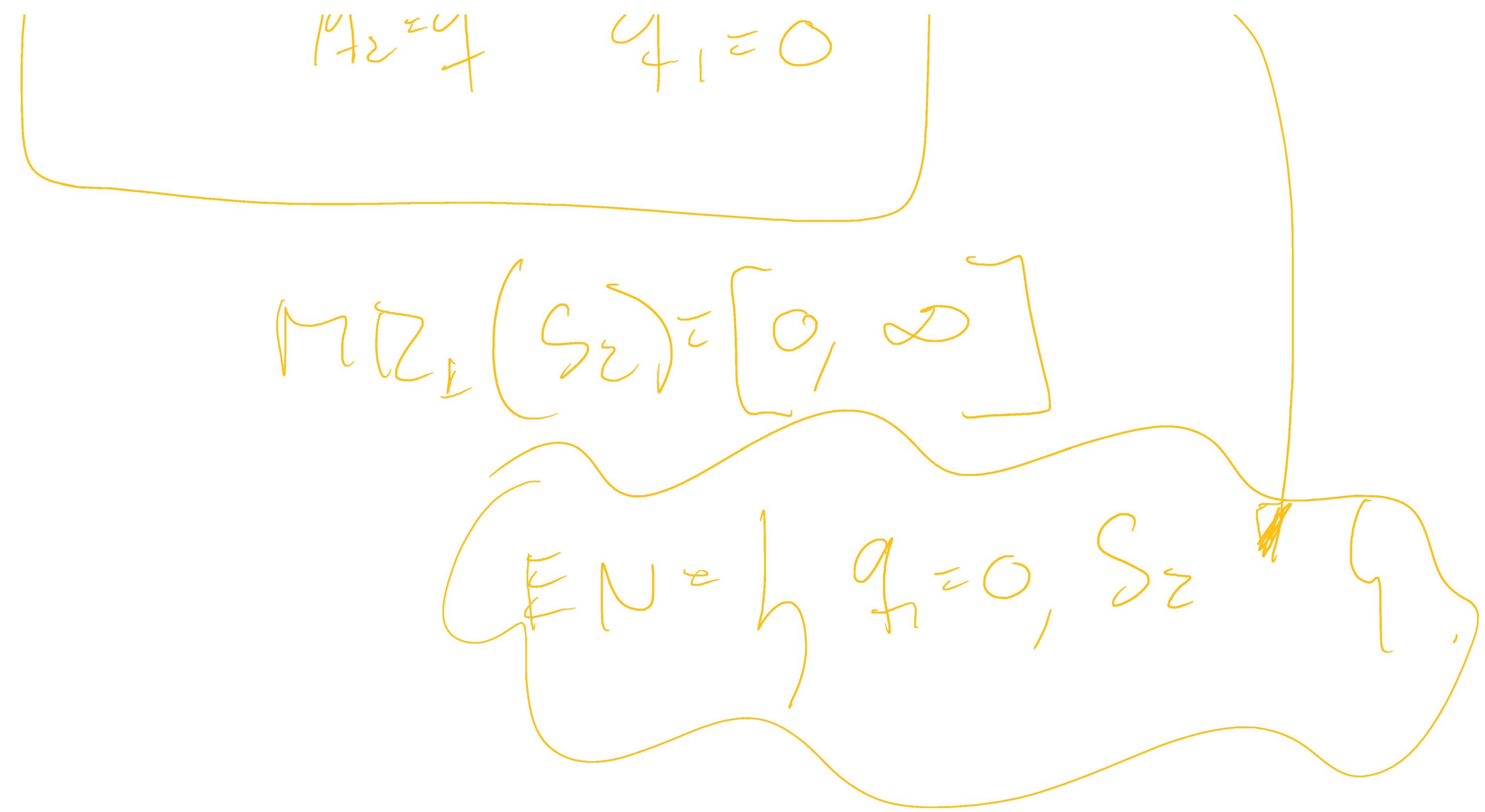
- The utility function of firm 2 does not depend at all on what it chooses for $q_2^*(q_1)$ when $q_1 \neq \alpha$
- In particular, q_2^* is a best response for firm 2

- The above observation allows us to conclude that there are many Nash equilibria of this game



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- In particular, in the Nash equilibrium corresponding to $\alpha = 0$, the equilibrium outcome is for firm 1 to choose a quantity of 0 and firm 2 setting a price of $A/2$
- This would be the same outcome if firm 2 were the monopolist in this market

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- Firm 2 is threatening to overproduce if firm 1 produces anything at all
- As a result, the best that firm 1 can do is to produce nothing
- If firm 1 were to hypothetically choose $q_1 > 0$, then firm 2 would obtain negative profits if it indeed follows through with $q_2^*(q_1)$.

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- To eliminate such counterintuitive equilibria, we focus instead on SPNE instead of NE
- Lets continue with the setting in which marginal costs are zero and the demand function is given by $A - q_1 - q_2$

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- The utility function of firm 2 is given by:

$$u_2(q_1, q_2(\cdot)) = (A - q_1 - q_2(q_1))q_2(q_1).$$

- So, player 2 solves:

$$\max_{q_2(\cdot)} (A - q_1 - q_2(q_1))q_2(q_1).$$

- **Case 1:** $q_1 > A$

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- **Case 1:** $q_1 > A$
- In this case, the best response of firm 2 is to set a quantity $q_2^*(q_1) = 0$ since producing at all gives negative profits.
- **Case 2:** $q_1 \leq A$
- In this case, the first order condition implies:

$$q_2^*(q_1) = \frac{A - q_1}{2}.$$

- Thus, in any SPNE, player 2 must play the following strategy:

$$q_2^*(q_1) = \begin{cases} \frac{A-q_1}{2} & \text{if } q_1 \leq A \\ 0 & \text{if } q_1 > A. \end{cases}$$

- Then player 1's utility function given that player 2 plays q_2^* is given by:

$$u_1(q_1, q_2^*(\cdot)) = q_1(A - q_1 - q_2^*(q_1)) = \begin{cases} q_1(A - q_1) & \text{if } q_1 > A, \\ q_1 \frac{A-q_1}{2} & \text{if } q_1 \leq A. \end{cases}$$

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- Thus, firm 1 maximizes:

$$\max_{q_1 \in [0,A]} q_1 \frac{A - q_1}{2}.$$

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- The **equilibrium outcome** is for firm 1 to choose $A/2$ and firm 2 to choose $A/4$

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- Lets solve for the set of SPNE (which is the same as NE) in the Cournot game with the same demand function and same costs
- In this case, (q_1^*, q_2^*) is a NE if and only if

$$q_1^* \in BR_1(q_2^*), q_2^* \in BR_2(q_1^*).$$

- For $q_1^* \in BR_1(q_2^*)$, we need q_1^* to solve the following maximization problem:

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- As a result, solving these two equations, we get:

$$q_1^* = q_2^* = \frac{A}{3}.$$

In the Cournot game, note that firms' payoffs are:

$$\pi_1^c = \frac{A^2}{9}, \pi_2^c = \frac{A^2}{9}.$$

As we already saw, this was not Pareto efficient since each firm is getting a payoff that is strictly less than 1/2 of the monopoly profits.

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- But by choosing something optimal, firm 1 will be able to do even better