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Lecture17....

Lecture 17: Applications of Subgame Perfect Nash Equilibrium

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Lecture 17: Repeated Games

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- We will use (G, T) to denote that game G is repeated T times

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2. Players observe the actions chosen by the players in period 1. Then in period 2, players simultaneously play the game G.
3. This game proceeds until time T.
4. After time T, if the action profiles chosen in times $1, 2, \ldots, T$ are given by $((a_i^1, a_{-i}^1), \ldots, (a_i^T, a_{-i}^T))$:

$$\sum_{t=1}^{T} \delta^{t-1} u_i(a_i^t, a_{-i}^t).$$

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- Working incurs a cost of 1 however increases the utility of the other player $-i$ by 2
- Thus,

$$u_i(e_i, e_{-i}) = 2e_{-i} - e_i.$$

Prisoner's Dilemma (Game G)

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

$NE = (e_1 = 0, e_2 = 0)$

- What happens when $T = 1$

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- NE: Players 1 and 2 will both choose $(e_1 = 0, e_1 = 0)$

$\delta = \frac{1}{1+r}$

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1. Both players play the simultaneous move game G.
2. Both players observe the actions chosen by the two players. Then they play G again.
3. Then payoffs are realized as the discounted sum of the utilities of the actions in each period with discount factor $\delta \in (0, 1]$.

Suppose that the two players chose $(e_1 = 1, e_2 = 1)$ in the first period
In the second period, they chose $(e_1 = 0, e_2 = 1)$

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$$u_1 = 1 + \delta \cdot 2$$
$$u_2 = 1 + \delta \cdot (-1).$$

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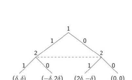
- Player 1 has 5 information sets in total
- A pure strategy for player 1 must specify what he does in each of these information sets
- Player 1 has a total of 32 (2^5) pure strategies
- Similarly, player 2 has a total of 32 pure strategies

- There are 5 subgames



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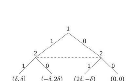
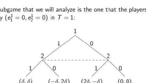
- Start at the end of the game (i.e., $T = 2$)



- There are 5 subgames
- Start at the end of the game (i.e., $T = 2$)
- The first subgame that we will analyze is the one that the players encounter after having play $(e_1^1 = 0, e_2^1 = 0)$ in $T = 1$:



The Nash equilibria can be seen by writing out the normal form of the game.

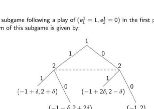
Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	δ, δ	$-\delta, 2\delta$
$e_1 = 0$	$2\delta, -\delta$	$0, 0$

- This game has a unique Nash equilibrium in which the players play $(e_1^2 = 0, e_2^2 = 0)$

- This game has a unique Nash equilibrium in which the players play $(e_1^2 = 0, e_2^2 = 0)$
- Therefore after having observed $(e_1^1 = 0, e_2^1 = 0)$ in the first period, both players will play $(e_1^2 = 0, e_2^2 = 0)$ in period 2

Consider the subgame following a play of $(e_1^1 = 1, e_2^1 = 0)$ in the first period. The extensive form of this subgame is given by:



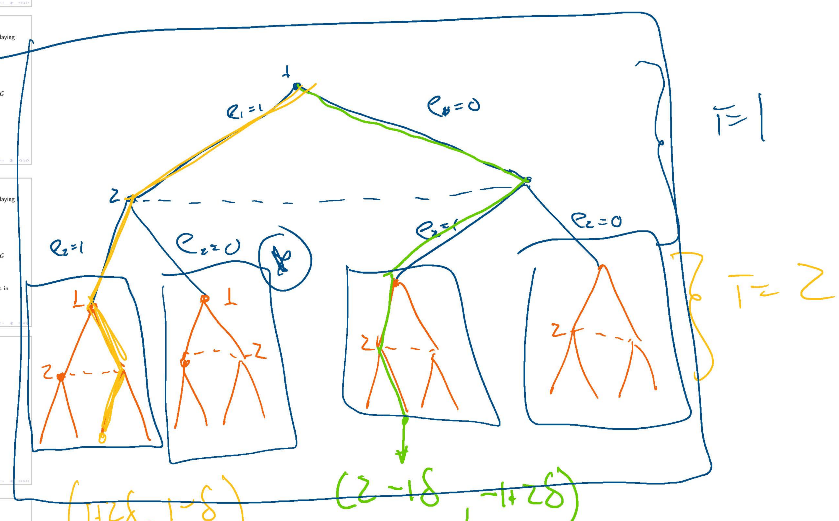
The normal form of this subgame can be seen in the Table

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1+\delta, 2+\delta$	$-1-\delta, 2+2\delta$
$e_1 = 0$	$-1+2\delta, 2-\delta$	$-1, 2$

- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium

$0 - \frac{1}{1+r}$

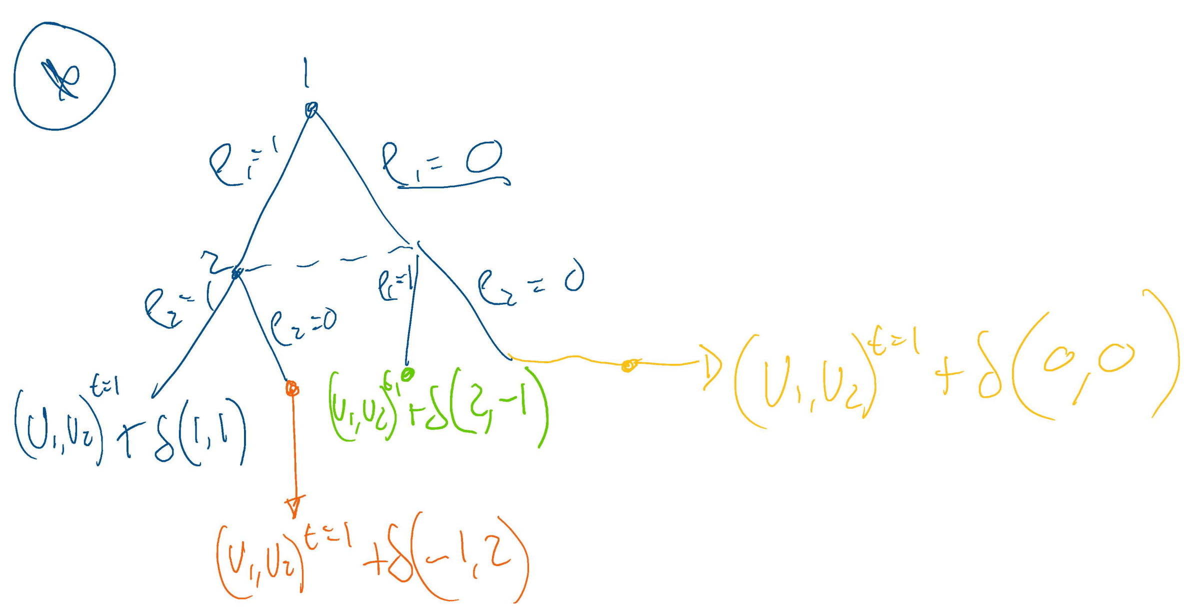


$(1+2\delta, 1-\delta)$ $(2-1\delta, -1+2\delta)$

$|S_1| = |(\ ,\ ,\ ,\ ,\ )| = 2^5 = 32$

$2 \times 2 \times 2 \times 2 \times 2$

$|S_2| = 2^5 = 32$

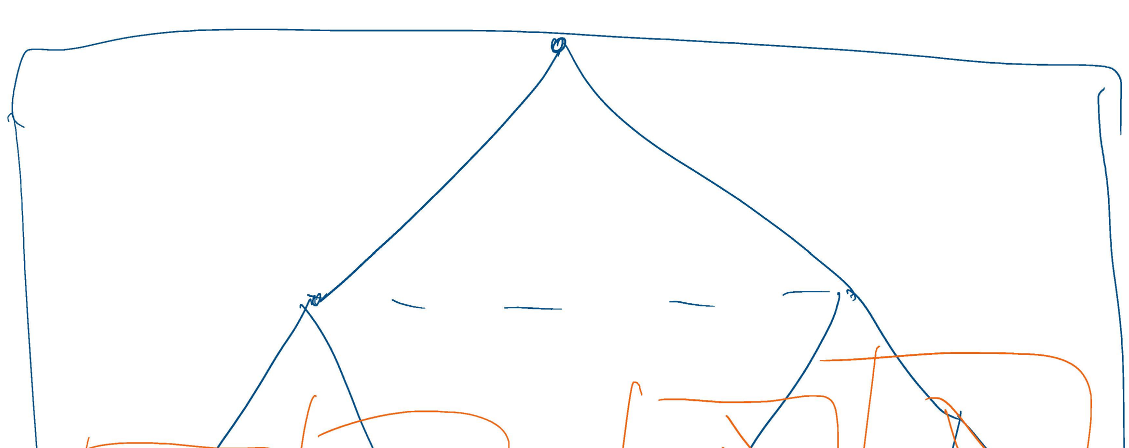


	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$(U_1, U_2)^{t=1} + \delta(1,1)$	$(U_1, U_2)^{t=1} + \delta(-1,2)$
$e_1 = 0$	$(U_1, U_2)^{t=1} + \delta(2,-1)$	$(U_1, U_2)^{t=1} + \delta(0,0)$

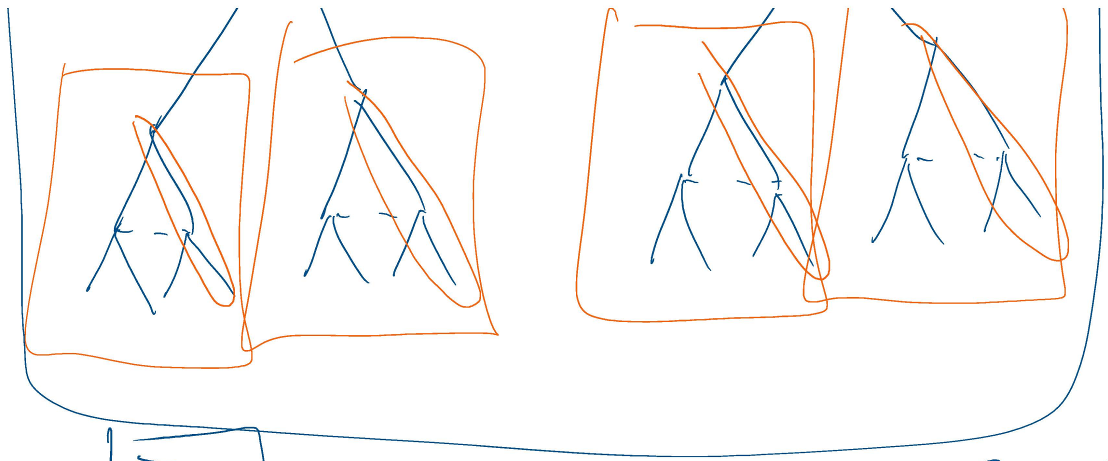
$(U_1, U_2)^{t=1} + \delta$

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

$\Longrightarrow$ EN. SUBJUEGOS DE $T = 2$ ES $(e_1 = 0, e_2 = 0)$



- $(e_1 = 0, e_2 = 0)$ is the unique Nash equilibrium
- In any SPNE, $(e_1^2 = 0, e_2^2 = 0)$ must be played after observing $(e_1^1 = 1, e_2^1 = 0)$



$\boxed{T=1}$

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	1, 1	-1, 2
$e_1 = 0$	2, -1	0, 0

$+ \ \delta(0,0)$

$EN \Rightarrow (e_1 = 0, \ e_2 = 0)$

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames

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- We will reach the same conclusion in each of these scenarios: that $(e_1^2 = 0, e_2^2 = 0)$ must be played in each of these subgames
- Regardless of the observed action, $(0, 0)$ is played in period 2
- Why is this the case?
- The idea is that payoffs that have accrued in period 1 are essentially sunk, and have no influence on incentives in period 2

To see this consider the normal form representation in the subgame after the observation of $(e_1^1 = 1, e_2^1 = 0)$

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$-1 + \delta, 2 + \delta$	$-1 - 1\delta, 2 + 2\delta$
$e_1 = 0$	$-1 + 2\delta, 2 - \delta$	$-1, 2$

- We can subtract off the payoff that player 1 received in period 1 and divide through player 1's payoffs by δ to obtain the following payoff matrix

Normal Form of Extensive Form

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- We can do the same thing for player 2's payoffs and get the payoff matrix

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
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- This will be true no matter the action profile played in period 1

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- Basically after any history, the strategic normal form is essentially the same as the original prisoner's dilemma
- Both players play $(e_1^2 = 0, e_2^2 = 0)$ after any information set in the last period

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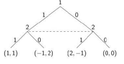
- We can simplify the extensive form game to the following:



If we draw the normal form of this game, then we get:

Normal Form of Extensive Form

	$e_2 = 1$	$e_2 = 0$
$e_1 = 1$	$1, 1$	$-1, 2$
$e_1 = 0$	$2, -1$	$0, 0$

The unique Nash equilibrium of the above normal form game is $(e_1^1 = 0, e_2^1 = 0)$

Therefore the unique SPNE is:

$$\left(\left(e_1^1 = 0 \;\; \begin{matrix} e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \\ e_1^2 = 0 \end{matrix}\right), \left(e_2^1 = 0 \;\; \begin{matrix} e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \\ e_2^2 = 0 \end{matrix}\right)\right)$$

In other words both players always shirk

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- Whenever the stage game has a unique NE, then the only SPNE of a **finite horizon** repeated game with that stage game is the repetition of the stage game NE

Theorem

Suppose that the stage game G has exactly one NE, $(a_1^*, a_2^*, \ldots, a_n^*)$. Then for any $\delta \in (0, 1]$ and any T, the T-times repeated game has a unique SPNE in which all players i play a_i^* at all information sets.

"EN TODOS LOS PERIODOS"

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- We concentrate just on the payoffs in the future. Thus in period $T-1$, player i simply wants to maximize:

$$\max_{a_i \in A_i} \delta^{T-2} u_i(a_i, a_{-i}^{T-1}) + \delta^{T-1} u_i(a^*).$$

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- Thus again, for this to be a Nash equilibrium, we need $a_1^{T-1} = a_1^*, \ldots, a_n^{T-1} = a_n^*$.
- Following exactly this induction, we can conclude that every player must play a_i^* at all times and all histories