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Lecture19...

Lecture 19: Infinitely Repeated Games

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Introduction to Infinitely Repeated Games

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Introduction to Infinitely Repeated Games

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- This happened because there was a last period from which we could induct backwards (and there was a domino effect!)

- When the game is instead **infinitely** repeated, this argument no longer applies since there is no such thing as a last period

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- Then play moves to period $t + 1$ and the game continues in the same manner.

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- We can represent each information set of player i by a history:

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- We denote the set of all histories at time t as H^t

Prisoner's Dilemma

	C_2	D_2
C_1	$1,1$	$-1,2$
D_1	$2,-1$	$0,0$

- For example, if the stage game is the prisoner's dilemma, at period 1, there are 4 possible histories:

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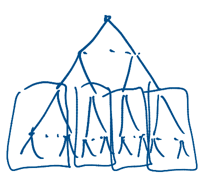
- This means that there is a **one-to-one** mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree
- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player i in each time t



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- Therefore, it is a function that describes:

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- Intuitively, s_i describes exactly what player i would do at every possible history h^t, where $s_i(h^t)$ describes what player i would do at history h^t

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- There can be more complicated strategies such as the following:

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- The above is called a **grim trigger strategy**

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- Intuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

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- However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.
- Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain

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- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \dots)$
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

	C_2	D_2
C_1	1,1	-1,2
D_1	2,-1	0,0

$$U_1(S_1, S_2) = \sum_{t=0}^{\infty} 1 \cdot \delta^t$$
$$= 1 + \delta + \delta^2 + \delta^3 + \dots$$
$$= \frac{1}{1-\delta}$$

$$S_T = 1 + \delta + \delta^2 + \dots + \delta^T$$
$$-\delta S_T = -\delta - \delta^2 - \delta^3 - \dots - \delta^{T+1}$$
$$S_T - \delta S_T = 1 - \delta^{T+1}$$
$$S_T(1-\delta) = 1 - \delta^{T+1}$$
$$S_T = \frac{1-\delta^{T+1}}{1-\delta}$$
$$\lim_{T \to \infty} S_T = \boxed{\frac{1}{1-\delta}}$$

S_1, S_2 es gatillo

$$U_1(S_1, S_2) = 1 + 1\delta + \delta^2 + \dots$$
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- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
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- Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta - 1}{1-\delta^2}.$$

$t=0 \to (C_1, D_2)$

$t=1 \to (D_1, C_2)$

$t=2 \to (C_1, D_2)$

$\vdots$

$$U_1(s_1, s_2) = -1 + 2\delta + -1\delta^2 + 2\delta^3 + \dots$$

$$= -1 - \delta^2 - \delta^4 - \delta^6 \dots = (-1)(1 + \delta^2 + \delta^4 + \delta^6) = (-1)\sum_{t=0}^{\infty} (\delta^2)^t = \frac{-1}{1-\delta^2}$$

$$+ 2\delta + 2\delta^3 + 2\delta^5 + \dots = +2\delta(1 + \delta^2 + \delta^4 \dots) = 2\delta \sum_{t=0}^{\infty} (\delta^2)^t = \frac{2\delta}{1-\delta^2}$$

$$= \frac{2\delta - 1}{1 - \delta^2}$$

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Introduction to Infinitely Repeated Games

Subgame Perfect Nash Equilibrium

Examples

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- What is a subgame perfect Nash equilibrium in an infinitely repeated game?
- It is exactly the same idea as in the finitely repeated game or more generally extensive form games
- That is a strategy profile $s = (s_1, \dots, s_n)$ is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s_i^t from time $t+1$ on

- This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s_i'.

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- We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

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- Why is this a SPNE?
- We can use the one-stage deviation principle

	C_2	D_2
C_1	1, 1	-1, 2
D_1	2, -1	$\boxed{0, 0}$

E.N!

T CUALQUIERA

$(1) + \cancel{\sum u_t} + \sum^{\infty} U_t$

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- Under this strategy profile s_1^*, s_2^*, for all histories h^t,
$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

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- Thus, for all histories h^t,
$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

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- Thus, (s_1^*, s_2^*) is a SPNE

In fact this is not specific to the prisoner's dilemma as we show below:

Theorem
Let a^* be a Nash equilibrium of the stage game. Then the strategy profile s^* in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

- What other kinds of SPNE are there?

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- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium

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$U\left(\cancel{\sum_{t=0}^{T-1} U_t}\right) + \sum_{t=0}^{\infty} U_t$

$U(ND) = \sum_{t=0}^{\infty} 0 \cdot \delta^t = 0$

$U(DESV) = U(C_1, D_2) + \sum_{t=1}^{\infty} \delta^t 0$

$= -1 + 0 = -1$

- What other kinds of SPNE are there?
- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium
- When the repeated game is infinitely repeated, this is no longer true

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- Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \ldots, C) \\ D_i & \text{if } h^t \neq (C, C, \ldots, C). \end{cases}$$

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- We will show that if δ is sufficiently high, so that the players are sufficiently patient, the strategy profile of grim trigger strategies is indeed a SPNE
- The *equilibrium path of play* for this SPNE is for players to play C in every period

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- How do we show that the above is indeed an SPNE?

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- We use the one-stage deviation principle again
- We need to check the one-stage deviation principle at every history h^t.

"SIN DESV ANTES"

T

$$U(ND) = \sum_{t=0}^{\infty} 1\,\delta^t = \frac{1}{1-\delta}$$

$$U(Desv) = U(D,C) + \sum_{t=1}^{\infty} \cancel{U(D,D)}\,\delta^t$$

0

$$= 2 + 0$$

$$U(ND) > U(Desv)$$

$$\frac{1}{1-\delta} > 2$$

$$\frac{1}{2} > 1-\delta$$

$$\boxed{\delta > \frac{1}{2}}$$

T "YA HUBO DESV"

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- But this is satisfied since D is a Nash equilibrium of the stage game

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- Therefore, the above is satisfied if and only if

$$1 + \delta \geq 2 \iff \delta \geq 1/2.$$

- Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \geq 1/2$.

- The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^* is a Nash equilibrium of the stage game. Suppose that $\hat{a}$ is an action profile of the Nash equilibrium such that

$$u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).$$

Then there is some $\delta^* < 1$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play $\hat{a}$ in every period.