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Thursday, May 5, 2022 2:28 PM

Lecture19...

Lecture 19: Infinitely Repeated Games

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Introduction to Infinitely Repeated Games

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- One of the features of **finitely** repeated games was that if the stage game had a **unique** Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium

- This happened because there was a last period from which we could induct backwards (and there was a domino effect!)

- When the game is instead **infinitely** repeated, this argument no longer applies since there is no such thing as a last period

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- Then play moves to period $t + 1$ and the game continues in the same manner.

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- We can represent each information set of player i by a history:

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- We denote the set of all histories at time t as H^t

Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$0, 0$

$T = 2$

$t = 2$

(C_1, C_2)
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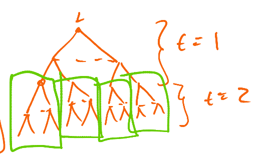

- For example, if the stage game is the prisoner's dilemma, at period 1, there are 4 possible histories:

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- This means that there is a **one-to-one** mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree

- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player i in each time t

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- Therefore, it is a function that describes:

$$s_i : \bigcup_{t \geq 0} H^t \to A_i.$$

- Intuitively, s_i describes exactly what player i would do at every possible history h^t, where $s_i(h^t)$ describes what player i would do at history h^t

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- There can be more complicated strategies such as the following:

$$s_i(h^t) = \begin{cases} C_i & \text{if } t = 0 \text{ or } h^t = (C, C, \ldots, C), \\ D_i & \text{otherwise.} \end{cases}$$

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- The above is called a **grim trigger strategy**

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- Then the payoff of player i in this repeated game is given by:
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- Intuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

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- However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.
- Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain

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$$U_1(C_1,C_2) + U_1(C_1,C_2)\delta + \dots = \sum_{t=0}^{\infty} U(C_1,C_2)\delta^t$$
$$= \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

$$\sum_{t=0}^{T} = 1 + \delta + \delta^2 + \dots + \delta^T$$
$$-\delta \sum_{t=0}^{T} = -\delta - \delta^2 - \dots - \delta^{T+1}$$
$$\sum_{t=0}^{T} \delta^t - \delta \sum_{t=0}^{T} \delta^t = 1 - \delta^{T+1}$$

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$$\sum_{t=0}^{T} \delta^t (1-\delta) = 1 - \delta^{T+1}$$

$$\sum_{t=0}^{T} \delta^t \quad \frac{1-\delta^{T+1}}{1-\delta}$$

$$\boxed{\lim_{T \to \infty} \sum_{t=0}^{T} \delta^t = \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}}$$

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- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \dots)$

"Gatillo"

$$U_1(C_1, C_2) + U_1(C_1, C_2)\delta + U_1(C_1, C_2)\delta^2 + \dots$$

$$\sum_{t=0}^{\infty} 1\,\delta^t = \frac{1}{1-\delta}$$

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- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again $(C, C, \dots)$
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

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- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
- Then if both players play these strategies, then the sequence of actions that arise is:

$$\underbrace{(C_1, D_2)}_{t=0}, \underbrace{(D_1, C_2)}_{t=1}, \underbrace{(C_1, D_2)}_{t=2}, \dots$$

$t \to 0 \to U_1(C_1, D_2) = -1$

$t \to 1 \to U_1(D_1, C_2) = 2$

$t \to 2 \to U_1(C_1, D_2) = -1$

$\vdots$

$$\sum U_1^t \delta^t = -1 + 2\delta - 1\delta^2 + 2\delta^3 - 1\delta^4 + \dots$$

$$= -1 - \delta^2 - \delta^4 - \delta^6 \dots$$
$$+ 2\delta + 2\delta^3 + 2\delta^5 + \dots$$

$$= (-1)(1 + \delta^2 + \delta^4 + \delta^6 \dots)$$
$$+ 2\delta(1 + \delta^2 + \delta^4 + \delta^6 \dots)$$

$$= \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta - 1}{1-\delta^2}$$

- How about a more complicated strategy profile?
- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
- Then if both players play these strategies, then the sequence of actions that arise is:

$$(C_1, D_2), (D_1, C_2), (C_1, D_2), \dots$$

- Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta - 1}{1-\delta^2}.$$

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- What is a subgame perfect Nash equilibrium in an infinitely repeated game?
- It is exactly the same idea as in the finitely repeated game or more generally extensive form games
- That is a strategy profile $s = (s_1, \ldots, s_n)$ is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s_i' from time $t+1$ on

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- This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s_i'.

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- We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

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- What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D_1 and D_2 always at all information sets
- Why is this a SPNE?
- We can use the one-stage deviation principle

$\boxed{T}$

$$U_1(\text{No Desv}) = U_1(D,D) + \sum_{t=1}^{\infty} U_1(D,D) = \sum_{t=0}^{\infty} 0\delta^t = 0$$

vs

$$U_1(\text{Desv}) = U_1(C,D) + \sum_{t=1}^{\infty} \cancel{U_1(D,D)} \to 0 = -1 + 0 = -1$$

Prisoner's Dilemma

	C_2	D_2
C_1	$1, 1$	$-1, 2$
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- Under this strategy profile s_1^*, s_2^*, for all histories h^t,
$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

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- Thus, for all histories h^t,
$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

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- Thus, (s_1^*, s_2^*) is a SPNE

In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^* be a Nash equilibrium of the stage game. Then the strategy profile s^* in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

- What other kinds of SPNE are there?

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- In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium
- When the repeated game is infinitely repeated, this is no longer true

- Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \ldots, C) \\ D_i & \text{if } h^t \neq (C, C, \ldots, C) \end{cases}$$

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- We will show that if δ is sufficiently high, so that the players are sufficiently patient, the strategy profile of grim trigger strategies is indeed a SPNE
- The *equilibrium path of play* for this SPNE is for players to play C in every period

- How do we show that the above is indeed an SPNE?

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- How do we show that the above is indeed an SPNE?
- We use the one-stage deviation principle again
- We need to check the one-stage deviation principle at every history h^t.

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- Suppose first that $h^t \neq (C, C, \ldots, C)$

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$$U(\text{No Desv}) = \sum_{t=0}^{\infty} U_1(C_1, C_2)\delta^t = \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

$$U(\text{Desv}) = U_1(D_1, C_2) + \sum_{t=1}^{\infty} U_1(D_1, D_2)\delta^t$$

$$= 2 + \boxed{\sum_{t=1}^{\infty} 0 \cdot \delta^t}$$

$$= 2 + \boxed{\frac{\delta}{1-\delta} \cdot 0}$$

$$= 2$$

$$U(ND) > U(D)$$

$$\frac{1}{1-\delta} > 2$$

$$\frac{1}{2} > 1 - \delta$$

$$\boxed{\delta > 1/2}$$

$$U(\text{No Desv}) = \sum_{t=0}^{\infty} U(D_1, D_2)\delta^t = 0$$

$$U(\text{Desv}) = U(C_1, D_2) + \cancel{\sum_{t=1}^{\infty} U(D_1, D_2)\delta^t} \rightarrow 0$$

$$= -1$$

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- Suppose first that $h^t \neq (C, C, \ldots, C)$
- Players are each suppose to play D_i
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$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\ \geq u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

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- But this is satisfied since D is a Nash equilibrium of the stage game

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- Therefore, the above is satisfied if and only if

$$1 + \delta \geq 2 \iff \delta \geq 1/2.$$

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- Players are both supposed to play C_i
- Thus, we need to check that

$$\begin{aligned} & u_i(C_i, C_{-i}) + \delta V_i(s^* \mid (h^t, C)) \\ & \geq u_i(D_i, C_{-i}) + \delta V_i(s^* \mid (h^t, (D_i, C_{-i}))). \end{aligned}$$

- In this case,

$$\begin{aligned} & V_i(s^* \mid (h^t, C)) = u_i(C_i, C_{-i}) \\ & = 1, V_i(s^* \mid (h^t, (D_i, C_{-i}))) = u_i(D) = 0. \end{aligned}$$

- Therefore, the above is satisfied if and only if

$$1 + \delta \geq 2 \iff \delta \geq 1/2.$$

- Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \geq 1/2$.

- The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^* is a Nash equilibrium of the stage game. Suppose that $\hat{a}$ is an action profile of the Nash equilibrium such that

$$u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).$$

Then there is some $\delta^* < 1$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play $\hat{a}$ in every period.