Lecture 19: Infinitely Repeated Games

Mauricio Romero

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games

One of the features of **finitely** repeated games was that if the stage game had a **unique** Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium

One of the features of **finitely** repeated games was that if the stage game had a unique Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium

This happened because there was a last period from which we could induct backwards (and there was a domino effect!) One of the features of **finitely** repeated games was that if the stage game had a unique Nash equilibrium, then the only subgame perfect Nash equilibrium was the repetition of that unique stage game Nash equilibrium

This happened because there was a last period from which we could induct backwards (and there was a domino effect!)

When the game is instead infinitely repeated, this argument no longer applies since there is no such thing as a last period

• We start with a stage game whose utilities are given by u_1, u_2, \ldots, u_n

• We start with a stage game whose utilities are given by u_1, u_2, \ldots, u_n

Each player *i* has an action set A_i



• We start with a stage game whose utilities are given by u_1, u_2, \ldots, u_n

Each player i has an action set A_i

▶ In each period t = 0, 1, 2, ..., players simultaneously choose an action $a_i \in A_i$ and the chosen action profile $(a_1, a_2, ..., a_n)$ is observed by all players

• We start with a stage game whose utilities are given by u_1, u_2, \ldots, u_n

Each player i has an action set A_i

▶ In each period t = 0, 1, 2, ..., players simultaneously choose an action $a_i \in A_i$ and the chosen action profile $(a_1, a_2, ..., a_n)$ is observed by all players

Then play moves to period t + 1 and the game continues in the same manner.

▶ It is impossible to draw the extensive form of this infinitely repeated game

▶ It is impossible to draw the extensive form of this infinitely repeated game

Each information set of each player *i* associated with a finitely repeated game corresponded to a history of action profiles chosen in the past

It is impossible to draw the extensive form of this infinitely repeated game

Each information set of each player *i* associated with a finitely repeated game corresponded to a history of action profiles chosen in the past

▶ We can represent each information set of player *i* by a history:

$$h^0 = (\emptyset), h^1 = (a^0 := (a^0_1, \dots, a^0_n)), \dots, h^t = (a^0, a^1, \dots, a^{t-1})$$

It is impossible to draw the extensive form of this infinitely repeated game

Each information set of each player *i* associated with a finitely repeated game corresponded to a history of action profiles chosen in the past

▶ We can represent each information set of player *i* by a history:

$$h^0 = (\emptyset), h^1 = (a^0 := (a^0_1, \dots, a^0_n)), \dots, h^t = (a^0, a^1, \dots, a^{t-1})$$

We denote the set of all histories at time t as H^t

Prisoner's Dilemma

	<i>C</i> ₂	D_2	
C_1	1,1	-1, 2	
D_1	2, -1	0,0	

 $\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$

◆□ ▶ ◆□ ▶ ◆ □ ▶ ◆ □ ▶ ● □ ● ● ● ●

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$$

For time t, H^t consists of 4^t possible histories

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$$

- For time t, H^t consists of 4^t possible histories
- This means that there is a one-to-one mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree

$$\{(C_1, C_2), (C_1, D_2), (D_1, C_2), (D_1, D_2)\} = H^1.$$

- For time t, H^t consists of 4^t possible histories
- This means that there is a one-to-one mapping between all possible histories and the information sets if we actually wrote out the whole extensive form game tree
- As a result, we can think of each $h^t \in H^t$ as representing a particular information set for each player *i* in each time *t*

It is simply a prescription of what player i would do at every information set or history

It is simply a prescription of what player i would do at every information set or history

▶ Therefore, it is a function that describes:

$$s_i: \bigcup_{t\geq 0} H^t \to A_i.$$

It is simply a prescription of what player i would do at every information set or history

Therefore, it is a function that describes:

$$s_i: \bigcup_{t\geq 0} H^t o A_i.$$

Intuitively, s_i describes exactly what player i would do at every possible history h^t, where s_i(h^t) describes what player i would do at history h^t

For example in the infinitely repeated prisoner's dilemma, the strategy s_i(h^t) = C_i for all h^t and all t is the strategy in which player i always plays C_i regardless of the history

For example in the infinitely repeated prisoner's dilemma, the strategy s_i(h^t) = C_i for all h^t and all t is the strategy in which player i always plays C_i regardless of the history

There can be more complicated strategies such as the following:

$$s_i(h^t) = egin{cases} C_i & ext{ if } t = 0 ext{ or } h^t = (C, C, \dots, C), \ D_i & ext{ otherwise.} \end{cases}$$

For example in the infinitely repeated prisoner's dilemma, the strategy s_i(h^t) = C_i for all h^t and all t is the strategy in which player i always plays C_i regardless of the history

There can be more complicated strategies such as the following:

$$s_i(h^t) = \begin{cases} C_i & \text{if } t = 0 \text{ or } h^t = (C, C, \dots, C), \\ D_i & \text{otherwise.} \end{cases}$$

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ の00

The above is called a grim trigger strategy

Suppose the strategies s₁,..., s_n are played which lead to the infinite sequence of action profiles:

 $a^0, a^1, \ldots, a^t, a^{t+1}, \ldots$

Suppose the strategies s₁,..., s_n are played which lead to the infinite sequence of action profiles:

$$a^0, a^1, \ldots, a^t, a^{t+1}, \ldots$$

▶ Then the payoff of player *i* in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t u_i(a^t).$$

◆□▶ ◆□▶ ◆目▶ ◆目▶ 目 のへの

Suppose the strategies s₁,..., s_n are played which lead to the infinite sequence of action profiles:

$$a^0, a^1, \ldots, a^t, a^{t+1}, \ldots$$

▶ Then the payoff of player *i* in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t u_i(a^t).$$

lntuitively, the contribution to payoff of time t action profile a^t is discounted by δ^t

It may be unreasonable to think about an infinitely repeated game

It may be unreasonable to think about an infinitely repeated game

However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.

It may be unreasonable to think about an infinitely repeated game

However the discount factor instead could be interpreted by the probability of the game/relationship ending at any point in time.

Thus, an infinitely repeated game does not necessarily represent a scenario in which there are an infinite number of periods, but rather a relationship which ends in finite time with probability one, but in which the time at which the relationship ends is uncertain Lets see some examples of how to compute payoffs in the repeated game

Lets see some examples of how to compute payoffs in the repeated game

• Consider first the strategy profile in which $s_i(h^t) = C_i$ for all i = 1, 2 and all h^t .

- Lets see some examples of how to compute payoffs in the repeated game
- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all i = 1, 2 and all h^t .
- ▶ In this case, the payoff of player 1 in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

- Lets see some examples of how to compute payoffs in the repeated game
- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all i = 1, 2 and all h^t .
- ▶ In this case, the payoff of player 1 in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

What about in the grim trigger strategy profile?

- Lets see some examples of how to compute payoffs in the repeated game
- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all i = 1, 2 and all h^t .
- ▶ In this case, the payoff of player 1 in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again (C, C, ...)

- Lets see some examples of how to compute payoffs in the repeated game
- Consider first the strategy profile in which $s_i(h^t) = C_i$ for all i = 1, 2 and all h^t .
- ▶ In this case, the payoff of player 1 in this repeated game is given by:

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

- What about in the grim trigger strategy profile?
- In that case, if all players play the grim trigger strategy profile, the sequence of actions that arise is again (C, C, ...)
- Thus the payoffs of all players is again $\frac{1}{1-\delta}$.

How about a more complicated strategy profile?

- How about a more complicated strategy profile?
- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period

- How about a more complicated strategy profile?
- Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period
- Then if both players play these strategies, then the sequence of actions that arise is:

 $(C_1, D_2), (D_1, C_2), (C_1, D_2), \ldots$



How about a more complicated strategy profile?

Suppose that $s_i(h^0) = (C_1, D_2)$ and the strategy profile says to do exactly what the opponent did in the previous period

Then if both players play these strategies, then the sequence of actions that arise is:

$$(C_1, D_2), (D_1, C_2), (C_1, D_2), \ldots$$

Then the payoff to player 1 in this game is given by:

$$\sum_{t=0}^{\infty} \delta^{2t}(-1) + \delta^{2t+1} \cdot 2 = \frac{-1}{1-\delta^2} + \frac{2\delta}{1-\delta^2} = \frac{2\delta-1}{1-\delta^2}.$$

◆□ > ◆□ > ◆ □ > ● □ >

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games Subgame Perfect Nash Equilibrium Examples

What is a subgame perfect Nash equilibrium in an infinitely repeated game?

<ロト < 団 ト < 臣 ト < 臣 ト 三 の < で</p>

What is a subgame perfect Nash equilibrium in an infinitely repeated game?

It is exactly the same idea as in the finitely repeated game or more generally extensive form games

What is a subgame perfect Nash equilibrium in an infinitely repeated game?

It is exactly the same idea as in the finitely repeated game or more generally extensive form games

That is a strategy profile s = (s₁,..., s_n) is a subgame perfect game Nash equilibrium if and only if s is a Nash equilibrium in every subgame of the repeated game.

Theorem (One-stage deviation principle)

s is a subgame perfect Nash equilibrium (SPNE) if and only if at every time t, and every history and every player i, player i cannot profit by deviating just at time t and following the strategy s'_i from time t + 1 on

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ の00

This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s'_i.

This is extremely useful since we only need to check that s_i is optimal against all possible one-stage deviations rather than having to check that it is optimal against all s'_i.

We will now put this into practice to analyze subgame perfect Nash equilibria of infinitely repeated games

Lecture 19: Infinitely Repeated Games

Introduction to Infinitely Repeated Games Subgame Perfect Nash Equilibrium Examples

◆□ > ◆□ > ◆ Ξ > ◆ Ξ > → Ξ = の < @

What is an example of a subgame perfect Nash equilibrium?

▶ What is an example of a subgame perfect Nash equilibrium?

One kind of equilibrium should be straightforward: each player plays D₁ and D₂ always at all information sets

- ▶ What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D₁ and D₂ always at all information sets

► Why is this a SPNE?

- ▶ What is an example of a subgame perfect Nash equilibrium?
- One kind of equilibrium should be straightforward: each player plays D₁ and D₂ always at all information sets

▲□▶ ▲□▶ ▲ □▶ ▲ □▶ ▲ □ ● ● ● ●

► Why is this a SPNE?

We can use the one-stage deviation principle

Prisoner's Dilemma

	<i>C</i> ₂	D_2	
C_1	1,1	-1, 2	
D_1	2, -1	0,0	

• Under this strategy profile s_1^*, s_2^* , for all histories h^t ,

$$V_1(s_1^*,s_2^*\mid h^t)=V_2(s_1^*,s_2^*\mid h^t)=0.$$

• Under this strategy profile s_1^*, s_2^* , for all histories h^t ,

$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

 \blacktriangleright Thus, for all histories h^t ,

$$\underbrace{u_i(D_i, D_{-i})}_0 + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_0 > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_0$$

• Under this strategy profile s_1^*, s_2^* , for all histories h^t ,

$$V_1(s_1^*, s_2^* \mid h^t) = V_2(s_1^*, s_2^* \mid h^t) = 0.$$

 \blacktriangleright Thus, for all histories h^t ,

$$\underbrace{u_i(D_i, D_{-i})}_{0} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0} > \underbrace{u_i(C_i, D_{-i})}_{-1} + \delta \underbrace{V_i(s_1^*, s_2^* \mid h^t)}_{0}$$

• Thus,
$$(s_1^*, s_2^*)$$
 is a SPNE

◆□ → ◆□ → ◆三 → ◆三 → ● ● ● ● ●

In fact this is not specific to the prisoner's dilemma as we show below:

Theorem

Let a^{*} be a Nash equilibrium of the stage game. Then the strategy profile s^{*} in which all players i play a_i^* at all information sets is a SPNE for any $\delta \in [0, 1)$.

What other kinds of SPNE are there?

What other kinds of SPNE are there?

In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium

What other kinds of SPNE are there?

In finitely repeated games, this was the only SPNE with prisoner's dilemma since the stage game had a unique Nash equilibrium

◆□▶ ◆□▶ ◆目▶ ◆目▶ 目 のへの

When the repeated game is infinitely repeated, this is no longer true

Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{ if } h^t = (C, C, \dots, C) \\ D_i & \text{ if } h^t \neq (C, C, \dots, C). \end{cases}$$

Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \dots, C) \\ D_i & \text{if } h^t \neq (C, C, \dots, C). \end{cases}$$

▲□▶ ▲□▶ ▲ 臣▶ ▲ 臣▶ 三臣 - めへぐ

We will show that if δ is sufficiently high, so that the players are sufficiently patient, the strategy profile of grim trigger strategies is indeed a SPNE

Consider for example the grim trigger strategy profile that we discussed earlier. Each player plays the following strategy:

$$s_i^*(h^t) = \begin{cases} C_i & \text{if } h^t = (C, C, \dots, C) \\ D_i & \text{if } h^t \neq (C, C, \dots, C). \end{cases}$$

We will show that if δ is sufficiently high, so that the players are sufficiently patient, the strategy profile of grim trigger strategies is indeed a SPNE

▶ The equilibrium path of play for this SPNE is for players to play C in every period

How do we show that the above is indeed an SPNE?

How do we show that the above is indeed an SPNE?

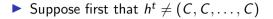
▶ We use the one-stage deviation principle again

How do we show that the above is indeed an SPNE?

We use the one-stage deviation principle again

• We need to check the one-stage deviation principle at every history h^t .

Case 1:



- Suppose first that $h^t \neq (C, C, \ldots, C)$
- ▶ Players are each suppose to play D_i

- Suppose first that $h^t \neq (C, C, \dots, C)$
- ▶ Players are each suppose to play D_i
- Thus, we need to check that

$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D))$$

 $\ge u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$

- Suppose first that $h^t \neq (C, C, \dots, C)$
- ▶ Players are each suppose to play D_i
- Thus, we need to check that

$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\ \geq u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

▶ But since
$$h^t \neq (C, C, ..., C)$$
,
 $V_i(s^* | (h^t, D)) = V_i(s^* | (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.

- Suppose first that $h^t \neq (C, C, \dots, C)$
- \blacktriangleright Players are each suppose to play D_i
- Thus, we need to check that

$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\ \ge u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

• But since
$$h^t \neq (C, C, ..., C)$$
,
 $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.

So the above inequality is satisfied if and only if

$$u_i(D_i, D_{-i}) \geq u_i(C_i, D_{-i}).$$

- Suppose first that $h^t \neq (C, C, \dots, C)$
- \blacktriangleright Players are each suppose to play D_i
- Thus, we need to check that

$$u_i(D_i, D_{-i}) + \delta V_i(s^* \mid (h^t, D)) \\ \ge u_i(C_i, D_{-i}) + \delta V_i(s^* \mid (h^t, (C_i, D_{-i})))$$

• But since $h^t \neq (C, C, ..., C)$, $V_i(s^* \mid (h^t, D)) = V_i(s^* \mid (h^t, (C_i, D_{-i}))) = u_i(D_i, D_{-i})$.

So the above inequality is satisfied if and only if

$$u_i(D_i, D_{-i}) \geq u_i(C_i, D_{-i}).$$

But this is satisfied since D is a Nash equilibrium of the stage game

Suppose instead that $h^t = (C, C, \dots, C)$

• Suppose instead that $h^t = (C, C, \dots, C)$

> Players are both supposed to play C_i

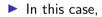
- Suppose instead that $h^t = (C, C, \dots, C)$
- > Players are both supposed to play C_i
- ► Thus, we need to check that

$$u_i(C_i, C_{-i}) + \delta V_i(s^* \mid (h^t, C)) \\ \ge u_i(D_i, C_{-i}) + \delta V_i(s^* \mid (h^t, (D_i, C_{-i}))).$$

- Suppose instead that $h^t = (C, C, \dots, C)$
- > Players are both supposed to play C_i
- ► Thus, we need to check that

$$u_i(C_i, C_{-i}) + \delta V_i(s^* \mid (h^t, C))$$

 $\geq u_i(D_i, C_{-i}) + \delta V_i(s^* \mid (h^t, (D_i, C_{-i}))).$



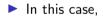
$$V_i(s^* \mid (h^t, C)) = u_i(C_i, C_{-i})$$

= 1, $V_i(s^* \mid (h^t, (D_i, C_{-i}))) = u_i(D) = 0.$

- Suppose instead that $h^t = (C, C, \dots, C)$
- > Players are both supposed to play C_i
- ▶ Thus, we need to check that

$$u_i(C_i, C_{-i}) + \delta V_i(s^* \mid (h^t, C))$$

 $\geq u_i(D_i, C_{-i}) + \delta V_i(s^* \mid (h^t, (D_i, C_{-i}))).$



$$V_i(s^* \mid (h^t, C)) = u_i(C_i, C_{-i})$$

= 1, $V_i(s^* \mid (h^t, (D_i, C_{-i}))) = u_i(D) = 0.$

Therefore, the above is satisfied if and only if

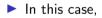
$$1 + \delta \ge 2 \iff \delta \ge 1/2.$$

・ロット (四)・ (目)・ (日)・ (日)

- Suppose instead that $h^t = (C, C, \dots, C)$
- > Players are both supposed to play C_i
- ► Thus, we need to check that

$$u_i(C_i, C_{-i}) + \delta V_i(s^* \mid (h^t, C))$$

 $\geq u_i(D_i, C_{-i}) + \delta V_i(s^* \mid (h^t, (D_i, C_{-i}))).$



$$V_i(s^* \mid (h^t, C)) = u_i(C_i, C_{-i})$$

= 1, $V_i(s^* \mid (h^t, (D_i, C_{-i}))) = u_i(D) = 0.$

Therefore, the above is satisfied if and only if

$$1 + \delta \ge 2 \iff \delta \ge 1/2.$$

▶ Thus the grim trigger strategy profile s^* is a SPNE if and only if $\delta \ge 1/2$.

The above findings that SPNE may involve the repetition of action profile that is not a stage game NE is not specific to just the infinitely repeated prisoner's dilemma as the following theorem demonstrates.

Theorem (Folk theorem)

Suppose that a^{*} is a Nash equilibrium of the stage game. Suppose that â is an action profile of the Nash equilibrium such that

$$u_1(\hat{a}) > u_1(a^*), \ldots, u_n(\hat{a}) > u_n(a^*).$$

Then there is some $\delta^* < 1$ such that whenever $\delta > \delta^*$, there is a SPNE in which on the equilibrium path of play, all players play \hat{a} in every period.