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Lecture 2: General Equilibrium

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Lecture 2: General Equilibrium

- Cobb-Douglas
- Using calculus
- Perfect substitutes
- Perfect complements

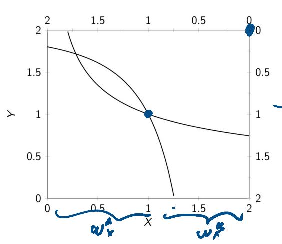
Cobb-Douglas

$$u_A(x, y) = x^{\alpha} y^{1-\alpha}$$
$$u_B(x, y) = x^{\beta} y^{1-\beta}$$

For graph suppose

$$\alpha = 0.7$$
$$\beta = 0.3$$
$$\boxed{\omega^A = (1, 1) \quad \omega^B = (1, 1)}$$

Cobb-Douglas



$$U_A(x,y) = x^{\alpha} y^{1-\alpha}$$

$$MRS^A := -\frac{\partial U/\partial x}{\partial U/\partial y} \quad \left(\frac{\frac{\text{utiles}}{x}}{\frac{\text{utiles}}{y}}\right) = \frac{y}{x}$$

$$MRS^A := -\frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha) x^{\alpha} y^{-\alpha}} = \frac{-\alpha}{1-\alpha} \frac{y_A}{x_A}$$

Cobb-Douglas

- Indifference curves must be tangent (formalize this later)
- Thus, the MRS must be equalized across the two consumers

$$MRS^A_{x,y} = \frac{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial x}}{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha} \frac{x^{\alpha-1} y^{1-\alpha}}{x^{\alpha} y^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{y^A}{x^A}$$

$$\boxed{MRS^B_{x,y}} = \frac{\frac{\partial x^{\beta} y^{1-\beta}}{\partial x}}{\frac{\partial x^{\beta} y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta} \frac{x^{\beta-1} y^{1-\beta}}{x^{\beta} y^{-\beta}} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

$$\boxed{\frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}}$$

Cobb-Douglas

But we haven't used the fact that

$$\boxed{\begin{aligned} x^A + x^B &= \omega_x = 2 \\ y^A + y^B &= \omega_y = 2 \end{aligned}}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\overbrace{\omega_y - y^A}^{y_B}}{\underbrace{\omega_x - x^A}_{x_B}}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{w_y - y^A}{w_x - x^A}$$

$$\boxed{\frac{y^A}{x^A} = \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y - y^A}{w_x - x^A}}$$

$$y^A = \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y - y^A}{w_x - x^A}\cdot x^A$$

$$= \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y x^A}{w_x - x^A} - \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{y^A x^A}{w_x - x^A}$$

$$y^A + \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{y^A x^A}{w_x - x^A} = \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y x^A}{w_x - x^A}$$

$$y^A\left(1 + \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{x^A}{w_x - x^A}\right) = \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y x^A}{w_x - x^A}$$

$$y^A = \frac{\frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{w_y x^A}{w_x - x^A}}{1 + \frac{(1-\alpha)\beta}{\alpha(1-\beta)}\frac{x^A}{w_x - x^A}}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{\omega_y}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

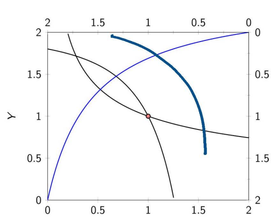
$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{\omega_y}{\omega_x - x^A}$$

Then:

$$y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}$$

Cobb-Douglas



$$\max_{x^A, y^A, x^B, y^B} U^A(x^A, y^A)$$

s.t.

$$U^B(x^B, y^B) \geq \overline{U} \rightarrow \text{O.P.}$$

$$\left.\begin{aligned} x^A + x^B &\leq w_x \\ y^A + y^B &\leq w_y \end{aligned}\right\} \text{FACTIBLE}$$

$$x^A, y^A, x^B, y^B \geq 0$$

Lecture 2: General Equilibrium

Cobb-Douglas

Using calculus

$$x^A, y^A, x^B, y^B > 0$$

Lecture 2: General Equilibrium

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Using calculus

Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$
$$u_B(x^B, y^B) \geq \underline{u}_B = u_B(x^{B*}, y^{B*})$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient if and only if it solves

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$
$$u_B(x^B, y^B) \geq \underline{u}_B$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\mathcal{L} = u_A(x^A, y^A) + \lambda(u_B(\underbrace{\omega_x - x^A}_{x^B}, \underbrace{\omega_y - y^A}_{y^B}) - \underline{u}_B)$$

$$\frac{\partial y}{\partial x^A} = \frac{\partial U_A}{\partial x_A} + \lambda \underbrace{\frac{\partial U_B}{\partial x_B}(-1)}_{\text{REGLA CADENA}} = 0$$

$$\frac{\partial y}{\partial y^A} = \frac{\partial U_A}{\partial y^A} + \lambda \frac{\partial U_B}{\partial y_B}(-1) = 0$$

$$\frac{\frac{\partial U_A}{\partial x_A}}{\frac{\partial U_A}{\partial y_A}} = \frac{\cancel{\lambda}}{\cancel{\lambda}} \frac{\partial U_B/\partial x_B}{\partial U_B/\partial y_B}$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A}: \frac{\partial u_A}{\partial x}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

For y^A:

$$\frac{\partial \mathcal{L}}{\partial y^A}: \frac{\partial u_A}{\partial y}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

If $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is an **interior** *feasible allocation. Then $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is Pareto efficient if and only if*

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > \underline{MRS^B_{x,y} = 1}$. Can we make both consumers better off?

$\frac{2y}{1x}$

$A \xrightarrow{1/2\, y} B$

$A \xleftarrow{1/2\, x} B$

MESON A

Y B INDIA

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- A gives up 1 unit of y to person B in exchange for unit of x
- B is indifferent since his $MRS^B_{x,y} = 1$.
- A receives a unit of x and only needs to give one unit of y (he was willing to give two)
- We have reallocated goods to make A strictly better off without hurting B

General case

$$\max_{((x_1^1,\dots,x_L^1),\dots,(x_1^I,\dots,x_L^I))} u_1(x_1^1,\dots,x_L^1) \text{ such that } u_2(x_1^2,\dots,x_L^2) \geq \underline{u}_2,$$
$$\vdots$$
$$u_I(x_1^I,\dots,x_L^I) \geq \underline{u}_I,$$
$$x_1^1 + \cdots + x_1^I \leq \omega_1,$$
$$\vdots$$
$$x_L^1 + \cdots + x_L^I \leq \omega_L.$$

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is a feasible interior allocation. Then $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is Pareto efficient if and only if $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ exhausts all resources **and** *for all pairs of goods ℓ, ℓ',*

$$MRS^1_{\ell,\ell'}(\hat{x}_1^1,\dots,\hat{x}_L^1) = \cdots = MRS^I_{\ell,\ell'}(\hat{x}_1^I,\dots,\hat{x}_L^I).$$

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

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Perfect substitutes

Suppose that

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$u_B(x^B, y^B) = x^B + y^B$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

$\bar{U}_A = 2x^A + y^A$

$y^A = \bar{U} - 2x^A$

$\Rightarrow \bar{U}_B = x^B + y^B$

$\boxed{y^B = \bar{U} - x_B} \Rightarrow \underbrace{w_y - y^A}_{y^B} = \bar{U} - (\underbrace{w_x - x_A}_{x_B})$

$\boxed{w_y - \bar{U} + w_x - x_A = y^A}$

$\boxed{4 - \bar{U} - x_A = y^A}$

MAX U^A S.E $U^B \geq \bar{U}_B$

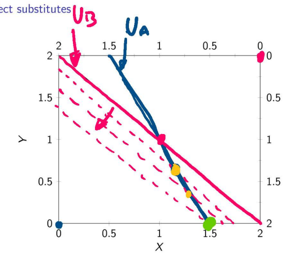
FACTIBLE

EQUIV

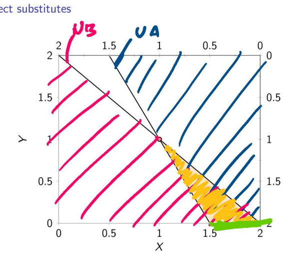
MAX U^B S.E $U^A \geq \bar{U}_A$

FACTIBLE

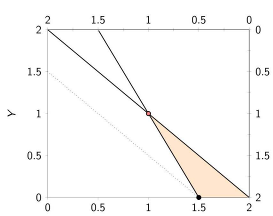
Perfect substitutes



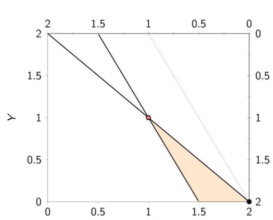
Perfect substitutes

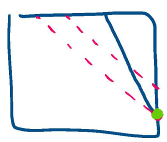


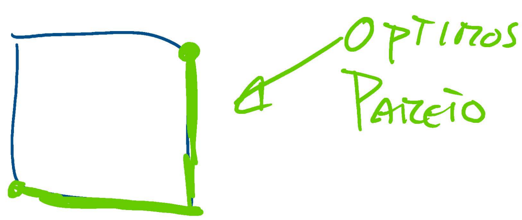
Perfect substitutes



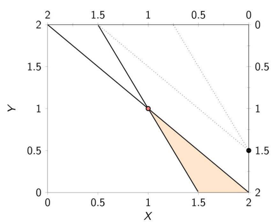
Perfect substitutes







Perfect substitutes



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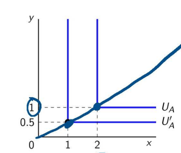
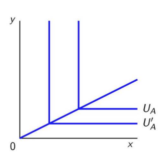
Perfect complements

Suppose that

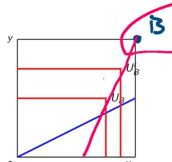
$$u_A(x^A, y^A) = \boxed{\min(x^A, 2y^A)} \rightarrow x^A = 2y^A$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (3, 1)$$
$$\omega^B = (1, 3)$$



SENDA EXPANCIÓN → $X_A = 2y_A$



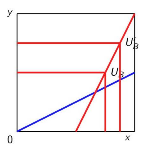
$U_B = \min(2X_B, y_B)$

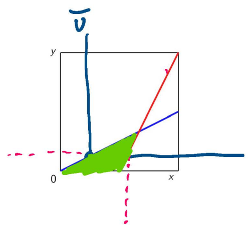


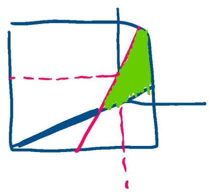
$\rightarrow 2X_B = y_B$

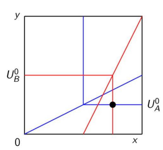
$2(\omega_x - X_A) = (\omega_y - y_A)$

$\rightarrow \underline{y_A = \omega_y - 2\omega_x + 2X_A}$

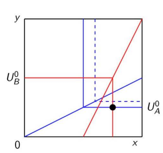




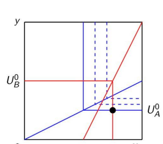




Make A as well as we can without making B worse off



Make A as well as we can without making B worse off



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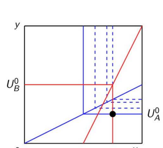

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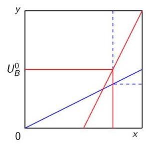

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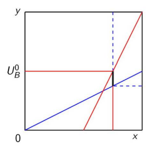
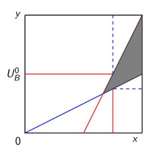
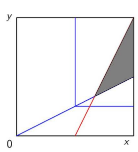
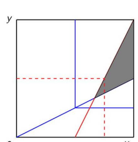
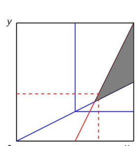


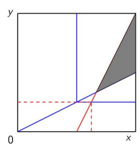
Make A as well as we can without making B worse off

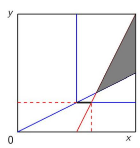


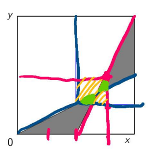


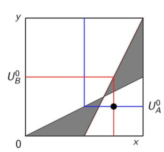


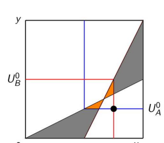












- What about: $u_A(x, y) = x^2 + y^2, u_B(x, y) = x + y$?
- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange box
- Can we say more? Not without prices