Lecture2.pdf

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Lecture 2: General Equilibrium

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Lecture 2: General Equilibrium

- Cobb-Douglas
- Using calculus
- Perfect substitutes
- Perfect complements

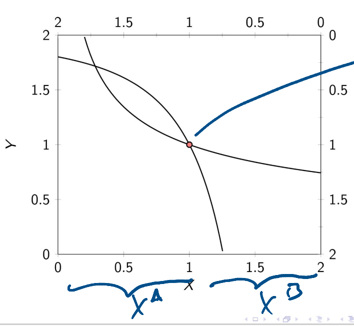
Cobb-Douglas

$$u_A(x,y) = x^{\alpha}y^{1-\alpha}$$
$$u_B(x,y) = x^{\beta}y^{1-\beta}$$

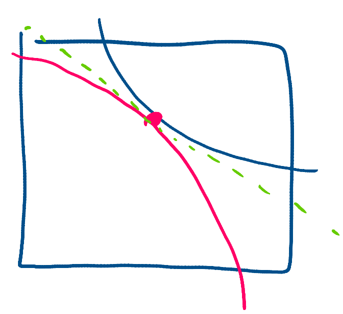
For graph suppose

$$\alpha = 0.7$$
$$\beta = 0.3$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

Cobb-Douglas



Pto Inicial



$TMS_A = TMS_B$

= utiles

Cobb-Douglas

Cobb-Douglas

- Indifference curves must be tangent (formalize this later)
- Thus, the MRS must be equalized across the two consumers

$$MRS^A_{x,y} = \frac{\frac{\partial x^\alpha y^{1-\alpha}}{\partial x}}{\frac{\partial x^\alpha y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha}\frac{x^{\alpha-1}y^{1-\alpha}}{x^\alpha y^{-\alpha}} = \frac{\alpha}{1-\alpha}\frac{y^A}{x^A}$$

$$MRS^B_{x,y} = \frac{\frac{\partial x^\beta y^{1-\beta}}{\partial x}}{\frac{\partial x^\beta y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta}\frac{x^{\beta-1}y^{1-\beta}}{x^\beta y^{-\beta}} = \frac{\beta}{1-\beta}\frac{y^B}{x^B}$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{y^B}{x^B}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$TMS_A = TMS_B$$

$$\frac{\partial U_A/\partial x_A}{\partial U_A/\partial y_A} = \frac{\frac{\text{utiles}}{x}}{\frac{\text{utiles}}{y}} = \boxed{\frac{y}{x}} \text{ UNIDADES}$$

$$\frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha)x^\alpha y^{-\alpha}} = \frac{\alpha}{1-\alpha}\frac{y_A}{x_A} = TMS^A$$

$$\frac{\beta}{1-\beta}\frac{y_B}{x_B} = TMS^B$$

$$\frac{\alpha}{1-\alpha}\frac{y_A}{x_A} = \frac{\beta}{1-\beta}\frac{y_B}{x_B} \quad \text{O.P.}$$

$$\frac{\alpha}{1-\alpha}\frac{y_A}{x_A} = \frac{\beta}{1-\beta}\frac{\overbrace{w_y - y_A}^{y_B}}{\underbrace{w_x - x_A}_{x_B}}$$

$$y_A = \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta} x_A \frac{w_y - y_A}{w_x - x_A}$$

$$y_A = \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A w_y}{w_x - x_A} - \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A y_A}{w_x - x_A}$$

$$y_A + \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A y_A}{w_x - x_A} = \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A w_y}{w_x - x_A}$$

$$y_A\left(1 + \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A}{w_x - x_A}\right) = \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A w_y}{w_x - x_A}$$

$$y_A = \frac{\frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A w_y}{w_x - x_A}}{1 + \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta}\frac{x_A}{w_x - x_A}}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

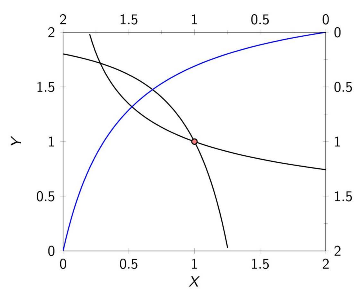
$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Then:

$$y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}$$

Cobb-Douglas



MAX $U_A(x_A, y_A)$
x_A, y_A, x_B, y_B

s.t

$U_B(x_B, y_B) \geq \bar{U}$

$x_A + x_B \leq \omega_x$

$y_A + y_B \leq \omega_y$

O.P.

Lecture 2: General Equilibrium

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Using calculus

Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \text{ such that}$$

$$\left.\begin{aligned} u_B(x^B, y^B) &\geq \underline{u}_B = u_B(x^{B*}, y^{B*}) \\ x^B + x^A &\leq \omega_x, \\ y^B + y^A &\leq \omega_y. \end{aligned}\right\} U_B(\omega_x - x_A, \omega_y - y_A) \geq \overline{U}$$

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A^*}, y^{A^*}, x^{B^*}, y^{B^*})$ is Pareto efficient if and only if it solves

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A, y^A) \textit{ such that}$$

$$\begin{aligned} u_B(x^B, y^B) &\geq \underline{u}_B \\ x^B + x^A &\leq \omega_x, \\ y^B + y^A &\leq \omega_y. \end{aligned}$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\max_{x^A, y^A} \mathcal{L} = u_A(x^A, y^A) + \lambda(u_B(\omega_x - x^A, \omega_y - y^A) - \underline{u}_B)$$

$$\frac{\partial \mathcal{L}}{\partial x_A} : \frac{\partial U_A}{\partial x_A} + \lambda \frac{\partial U_B}{\partial x_B}(-1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_A} : \frac{\partial U_A}{\partial y_A} + \lambda \frac{\partial U_B}{\partial y_B}(-1) = 0$$

$$\frac{\partial U_A/\partial x_A}{\partial U_A/\partial y_A} = \frac{\cancel{\lambda}}{\cancel{\lambda}} \frac{\partial U_B/\partial x_B}{\partial U_B/\partial y_B}$$

$$TMS^A = TMS^B$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A} : \frac{\partial u_A}{\partial x}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

For y^A:

$$\frac{\partial \mathcal{L}}{\partial y^A} : \frac{\partial u_A}{\partial y}(x^A, y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

If $(x^{A^*}, y^{A^*}, x^{B^*}, y^{B^*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A^*}, y^{A^*})}{\frac{\partial u_A}{\partial y}(x^{A^*}, y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A^*}, \omega_y - y^{A^*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A^*}, \omega_y - y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B^*}, y^{B^*})}{\frac{\partial u_B}{\partial y}(x^{B^*}, y^{B^*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A^*}, y^{A^*}, \omega_x - x^{A^*}, \omega_y - y^{A^*})$ is an **interior** *feasible allocation. Then $(x^{A^*}, y^{A^*}, \omega_x - x^{A^*}, \omega_y - y^{A^*})$ is Pareto efficient if and only if*

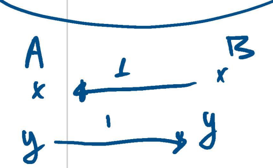
$$\frac{\frac{\partial u_A}{\partial x}(x^{A^*}, y^{A^*})}{\frac{\partial u_A}{\partial y}(x^{A^*}, y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A^*}, \omega_y - y^{A^*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A^*}, \omega_y - y^{A^*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B^*}, y^{B^*})}{\frac{\partial u_B}{\partial y}(x^{B^*}, y^{B^*})}.$$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

A: 2y for 1x

B: 1y for 1x



Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- A gives up 1 unit of y to person B in exchange for unit of x
- B is indifferent since his $MRS^B_{x,y} = 1$.
- A receives a unit of x and only needs to give one unit of y (he was willing to give two)
- We have reallocated goods to make A strictly better off without hurting B

General case

$$\max_{((x_1^1,\dots,x_L^1),\dots,(x_1^I,\dots,x_L^I))} u_1(x_1^1,\dots,x_L^1) \text{ such that } \begin{aligned} & u_2(x_1^2,\dots,x_L^2) \geq \underline{u}_2, \\ & \vdots \\ & u_I(x_1^I,\dots,x_L^I) \geq \underline{u}_I, \\ & x_1^1+\cdots+x_1^I \leq \omega_1, \\ & \vdots \\ & x_L^1+\cdots+x_L^I \leq \omega_L. \end{aligned}$$

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is a feasible interior allocation. Then $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is Pareto efficient if and only if $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ exhausts all resources **and** *for all pairs of goods ℓ,ℓ',*

$$MRS^1_{\ell,\ell'}(\hat{x}_1^1,\dots,\hat{x}_L^1) = \cdots = MRS^I_{\ell,\ell'}(\hat{x}_1^I,\dots,\hat{x}_L^I).$$

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

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Perfect substitutes

Suppose that

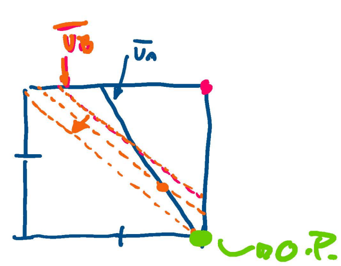
$$\boxed{u_A(x^A,y^A) = 2x^A + y^A}$$
$$\boxed{u_B(x^B,y^B) = x^B + y^B}$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

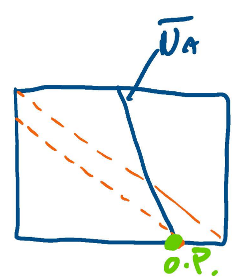
$\bar{u}_A = 2x^A + y^A$

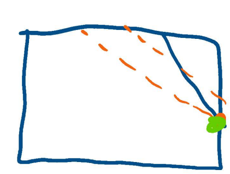
$\bar{u}_A - 2x^A = y^A$

$\bar{u}_B = x^B + y^B$

$\Rightarrow \bar{U}_B - (\omega_x - x_A) = \omega_y - y_A$







$\bar{u}_A = 2x^A + y^A$

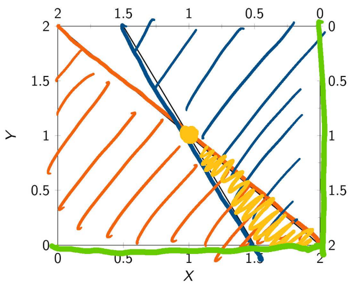
$\bar{u}_A - 2x^A = y^A$

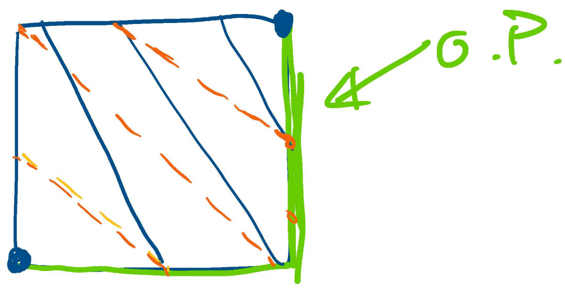
$\bar{U}_B = x^B + y^B$

$\boxed{\bar{U}_B - x^B = y^B} \Rightarrow \bar{U}_B - (w_x - X_A) = w_y - y_A$

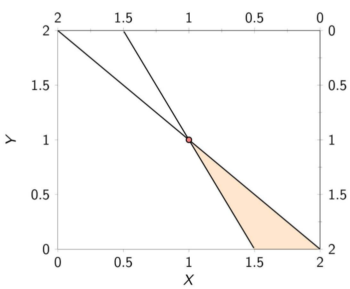
$y_A = \underbrace{w_y - \bar{U}_B + w_x}_{} - X_A$

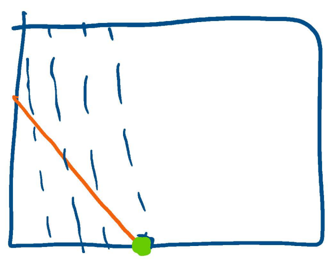
Perfect substitutes



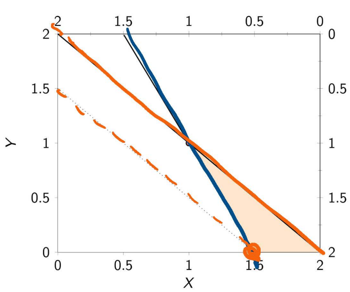


Perfect substitutes

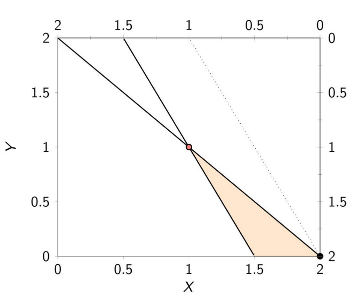




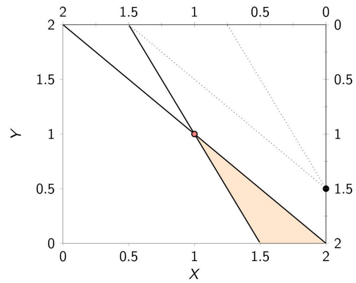
Perfect substitutes



Perfect substitutes



Perfect substitutes



Lecture 2: General Equilibrium

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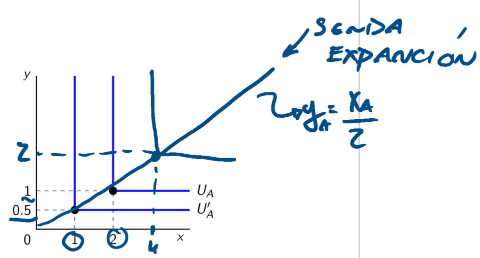
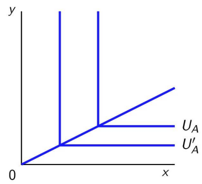
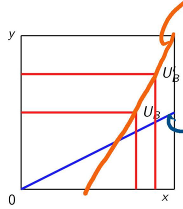
Perfect complements

Suppose that

$$u_A(x^A, y^A) = \min(x^A, 2y^A) \longrightarrow x^A = 2y^A$$
$$u_B(x^B, y^B) = \min(2x^B, y^B) \longrightarrow 2x^B = y^B$$
$$\omega^A = (3,1)$$
$$\omega^B = (1,3)$$



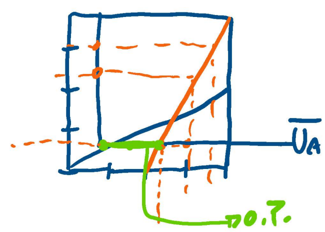


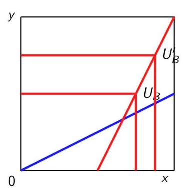


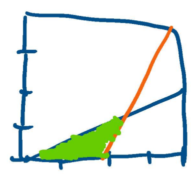
$\Rightarrow 2x^B = y^B \Rightarrow 2(w_x - x_A) = w_y - y_A$

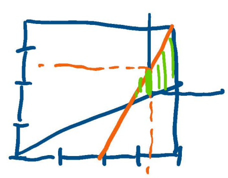
$y_A = w_y - 2w_x + 2x_A$

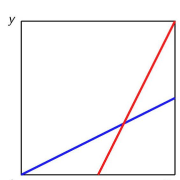
$\rightarrow y_A = \frac{x_A}{2}$

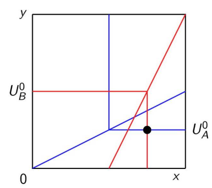




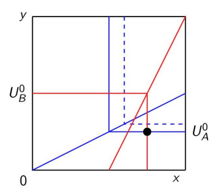








Make A as well as we can without making B worse off



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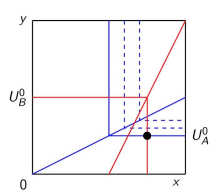

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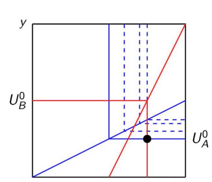

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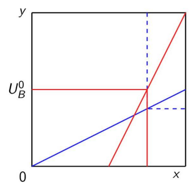

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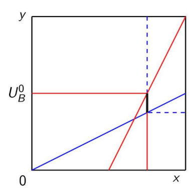
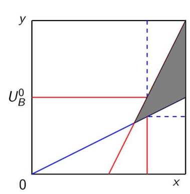
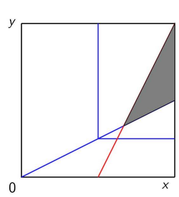
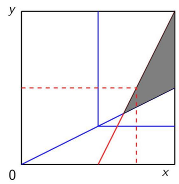
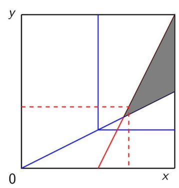


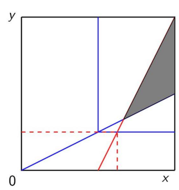
Make A as well as we can without making B worse off






y
0
x


y
0
x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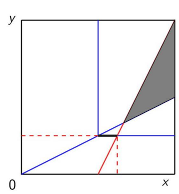
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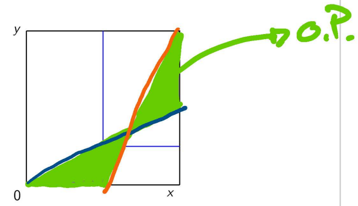

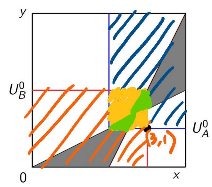

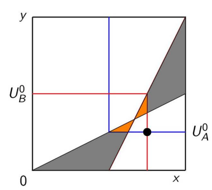

- What about: $u_A(x, y) = x^2 + y^2$, $u_B(x, y) = x + y$?

- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange box
- Can we say more? Not without prices