Lecture4

Thursday, February 3, 2022 11:50 AM

Lecture4

Lecture 4: General Equilibrium

Mauricio Romero

Lecture 4: General Equilibrium

- Is there always an equilibrium?
- Is the equilibrium unique?
- First welfare theorem
- Second welfare theorem

Lecture 4: General Equilibrium

- Is there always an equilibrium?
- Is the equilibrium unique?
- First welfare theorem
- Second welfare theorem

- The answer is going to be yes in general
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price is demand > supply), then the equilibrium is where this function stops updating

Lecture 4: General Equilibrium

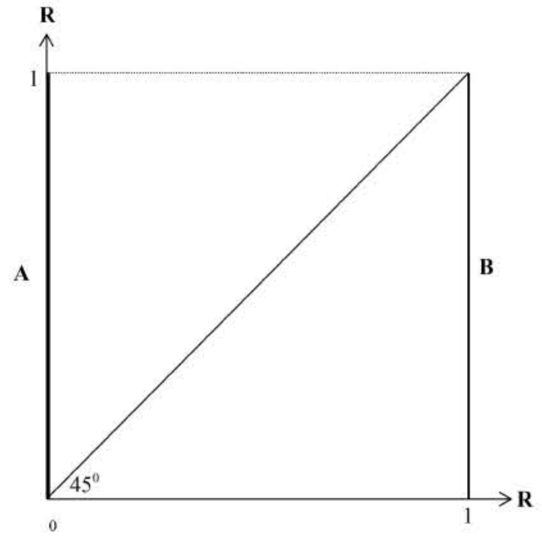
Is there always an equilibrium?
An intro to fix point theorems
The walrasian auctioneer

Is the equilibrium unique?

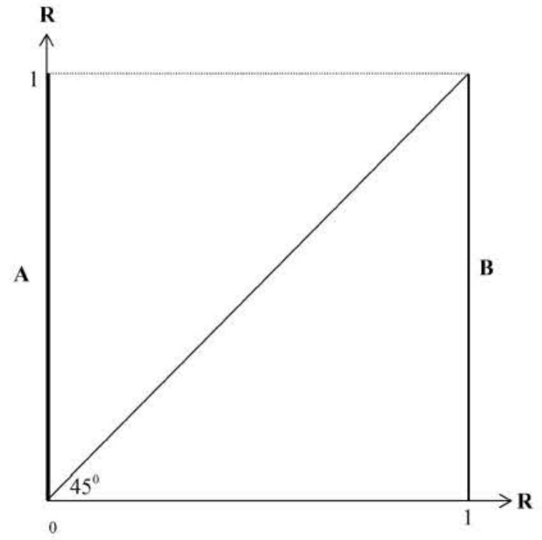
First welfare theorem

Second welfare theorem

Try to draw a line from A to B without crossing the diagonal

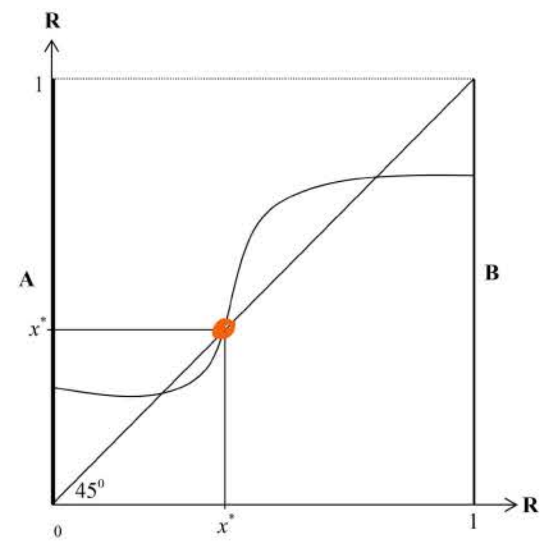


Try to draw a line from A to B without crossing the diagonal



Its impossible!

For example...



There is even a theorem for this:

Theorem

For any function $f : [0,1] \to [0,1]$ that is continuous, there exists an $x^* \in [0,1]$ such that $f(x^*) = x^*$

And a more general version!

Theorem

For any function $f : \triangle^{L-1} \to \triangle^{L-1}$ that is continuous, there exists a point $p^* = (p_1^*, p_2^*, ..., p_L^*)$ such that

$$f(p^*) = p^*.$$

where

$$\triangle^{L-1} = \{(p_1, p_2, ..., p_L) \in \mathbb{R}_+^L \mid \sum_{l=1}^{L} p_l = 1\}$$

What was the goal again?

- Prove the existence of a general equilibrium in a market
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price if demand > supply), then the equilibrium is where this function stops updating

Lecture 4: General Equilibrium

Is there always an equilibrium?

An intro to fix point theorems

The walrasian auctioneer

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Excess demand

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} x^{*i}(p) - \sum_{i=1}^{I} w^i$$

Excess demand

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} x^{*i}(p) - \sum_{i=1}^{I} w^i$$

since $x^{*i}(p)$ is the demand (i.e., consumers are already maximizing) then we have the following result:

vector

Remark

$p \in \mathbb{R}^L_{++}$ is a competitive equilibrium if and only if $Z(p) = 0$

Excess demand

Z(p) has the following properties

1. Is continuous in p
2. Is homogeneous of degree zero
3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law)

$$\boxed{\begin{gathered} P \cdot X_i = P \cdot W_i \\ \sum P X_i = \sum P \cdot W_i \\ P \sum (X_i - W_i) \\ P \cdot Z(P) = 0 \end{gathered}}$$

Excess demand

Z(p) has the following properties

1. Is continuous in p
2. Is homogeneous of degree zero
3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law) — Think about this!

Excess demand

We said we want to update prices in a "logical" way. If excess demand is positive, then increase prices...

Excess demand

We said we want to update prices in a "logical" way. If excess demand is positive, then increase prices...

$$\boxed{p' = p + Z(p)}$$

$$p' = p + \max(0, Z(p))$$

But what if $p' < 0$? Ok then

$$T(p) = \frac{1}{\sum_{l=1}^{L} p_l + max\,(0, Z_l(p))}\left(p_1 + \max(0, Z_1(p)),\right.$$
$$p_2 + \max(0, Z_2(p)), \ldots,$$
$$\left. p_L + \max(0, Z_L(p))\right)$$

$$p' = \frac{p + \max(0, Z(p))}{\sum_{i=1} p_i + \max(0, Z_i(p))}$$

$$T(p) = p'$$

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $\boxed{T(p^*) = p^*}$
- Then $Z(p^*) = 0$

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $T(p^*) = p^*$
- Then $Z(p^*) = 0$ (why?)

So when does it break down?

- We needed demand to be continuous!

Weird case - no equilibrium

$$\begin{aligned} u_A(x^A, y^A) &= \min(x^A, y^A) \\ u_B(x^B, y^B) &= \max(x^B, y^B) \\ \omega^A &= (1, 1) \\ \omega^B &= (1, 1) \end{aligned}$$

Weird case - no equilibrium

$$\begin{aligned} u_A(x^A, y^A) &= \min(x^A, y^A) \\ u_B(x^B, y^B) &= \max(x^B, y^B) \\ \omega^A &= (1, 1) \\ \omega^B &= (1, 1) \end{aligned}$$

- prices are positive (why?)

Weird case - no equilibrium

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, y^A)\\
u_B(x^B, y^B) &= \max(x^B, y^B)\\
\omega^A &= (1, 1)\\
\omega^B &= (1, 1)
\end{aligned}
$$

- prices are positive (why?)
- normalize $p_x = 1$

Weird case - no equilibrium

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, y^A)\\
u_B(x^B, y^B) &= \max(x^B, y^B)\\
\omega^A &= (1, 1)\\
\omega^B &= (1, 1)
\end{aligned}
$$

- prices are positive (why?)
- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible $Y^b = \frac{1}{p_y} + 1$

Weird case - no equilibrium

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, y^A)\\
u_B(x^B, y^B) &= \max(x^B, y^B)\\
\omega^A &= (1, 1)\\
\omega^B &= (1, 1)
\end{aligned}
$$

- prices are positive (why?)
- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible $Y^b = \frac{1}{p_y} + 1$
- if $p_y > 1$ then B wants to demand as much of x as possible $X^b = p_y + 1$

Weird case - no equilibrium

$$
\begin{aligned}
u_A(x^A, y^A) &= \min(x^A, y^A)\\
u_B(x^B, y^B) &= \max(x^B, y^B)\\
\omega^A &= (1, 1)\\
\omega^B &= (1, 1)
\end{aligned}
$$

- prices are positive (why?)
- normalize $p_x = 1$
- if $p_y < 1$ then B wants to demand as much of y as possible $Y^b = \frac{1}{p_y} + 1$
- if $p_y > 1$ then B wants to demand as much of x as possible $X^b = p_y + 1$
- if $p_y = 1$ then B either demands two units of X or two units of Y, but A demands at least one unit of each good

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Is the equilibrium unique?

We have seen it is not

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

First welfare theorem

Theorem

Consider any pure exchange economy. Suppose that all consumers have weakly monotone utility functions. Then if (x^*, p) is a competitive equilibrium, then x^* is a Pareto efficient allocation.

Proof

By contradiction:

Proof

By contradiction:
Assume that $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ is a competitive equilibrium but that $\left(x^1, x^2, ..., x^I\right)$ is not Pareto efficient

Proof

By contradiction:
Assume that $(p, (x^1, x^2, ..., x^I))$ is a competitive equilibrium but that $(x^1, x^2, ..., x^I)$ is not Pareto efficient
Then there is an allocation $(\hat{x}^1, \hat{x}^2, ..., \hat{x}^I)$ such that

- is feasible
- pareto dominates $(x^1, x^2, ..., x^I)$

Proof

By contradiction:
Assume that $(p, (x^1, x^2, ..., x^I))$ is a competitive equilibrium but that $(x^1, x^2, ..., x^I)$ is not Pareto efficient
Then there is an allocation $(\hat{x}^1, \hat{x}^2, ..., \hat{x}^I)$ such that

- is feasible
- pareto dominates $(x^1, x^2, ..., x^I)$

In other words:

1. $\boxed{\sum_{i=1}^I \hat{x}^i \leq \sum_{i=1}^I w^i}$
2. For all i, $u^i(\hat{x}^i) \geqslant u^i(x^i)$
3. For some i^*, $u^{i^*}(\hat{x}^{i^*}) > u^{i^*}(x^{i^*})$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \hat{x}^{i^*} > p \cdot w^{i^*}$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \hat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\hat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \hat{x}^i \geqslant p \cdot w^i$

$$\sum_{i=1}^{I} p \cdot x^i > \sum_{i=1}^{I} p \cdot w^i$$

$a \geq b$

$c > d$

$a + c > b + d$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \widehat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\widehat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \widehat{x}^i \geqslant p \cdot w^i$

Adding over all agents we get:

$$\sum_{i=1}^{I} p \cdot \widehat{x}^i > \sum_{i=1}^{I} p \cdot w^i$$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \widehat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\widehat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \widehat{x}^i \geqslant p \cdot w^i$

Adding over all agents we get:

$$\sum_{i=1}^{I} p \cdot \widehat{x}^i > \sum_{i=1}^{I} p \cdot w^i$$

Which in turn implies

$$p \cdot \sum_{i=1}^{I} \widehat{x}^i > p \cdot \sum_{i=1}^{I} w^i$$

Proof

By definition of an equilibrium we have that

- Condition 3 in the previous slide implies $p \cdot \widehat{x}^{i^*} > p \cdot w^{i^*}$
  - Otherwise, why didn't i^* pick $\widehat{x}^{i^*}$ to begin with
- Condition 2 in the previous slide implies that for all i, $p \cdot \widehat{x}^i \geqslant p \cdot w^i$

Adding over all agents we get:

$$\sum_{i=1}^{I} p \cdot \widehat{x}^i > \sum_{i=1}^{I} p \cdot w^i$$

Which in turn implies

$$p \cdot \sum_{i=1}^{I} \widehat{x}^i > p \cdot \sum_{i=1}^{I} w^i$$

$\sum \hat{x}^i > \sum w^i$

Which contradicts what Condition 1 in the previous slide implies.

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it

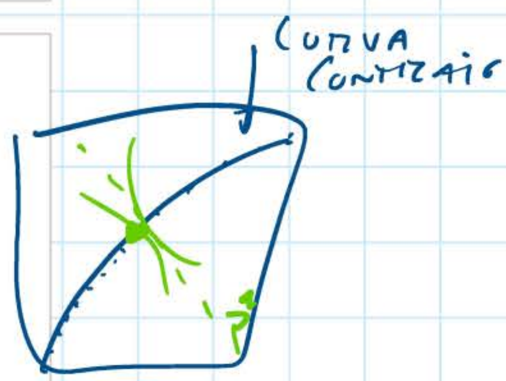
- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations
- How about the opposite?

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations
- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations
- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?

- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations
- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?
  - Not in general...



- Great! Since we motivated Pareto efficiency as the bare minimum, its nice to know that the market achieves it
- This may be useful in calculating competitive equilibrium... we only have to search within Pareto efficient allocations
- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?

  - Not in general... but what if we allow for a redistribution of resources?

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

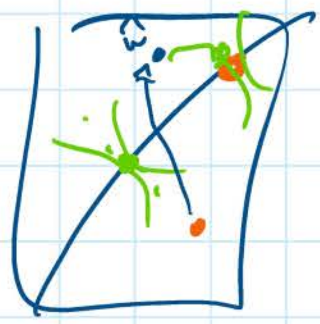
Second welfare theorem

Second welfare theorem

Theorem

Given an economy $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, w^i\right)_{i\in\mathcal{I}} \right\rangle$ where all consumers have weakly monotone, quasi-concave utility functions. If $\left(x^1, x^2, ..., x^I\right)$ is a Pareto optimal allocation then there exists a redistribution of resources $\left(\widehat{w}^1, \widehat{w}^2, ..., \widehat{w}^I\right)$ and some prices $p = (p_1, p_2, ..., p_L)$ such that:

1. $\sum_{i=1}^{I} \widehat{w}^i = \sum_{i=1}^{I} w^i$
2. $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ *is a competitive equilibrium of the economy* $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, \widehat{w}^i\right)_{i\in\mathcal{I}} \right\rangle$

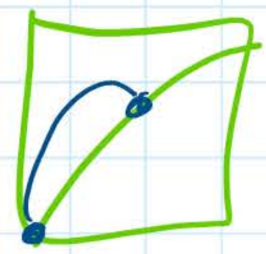


- Great, you don't need to close the markets to achieve a certain Pareto allocation

- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments

- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution?

- Great, you don't need to close the markets to achieve a certain Pareto allocation
- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution? Not taxes, since they produce dead-weight loss



- In contrast to the first welfare theorem, we require an additional assumption that all utility functions are quasi-concave.
- What if they are not? consider the following:

$$u_A(x,y) = \max\{x,y\}$$
$$u_B(x,y) = \min\{x,y\}$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

In this example, all points in the Edgeworth Box are Pareto efficient. However we cannot obtain any of these points as a competitive equilibrium after transfers.