Lecture6

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Lecture6

Lecture 6: General Equilibrium

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Lecture 6: General Equilibrium

A few things I forgot to say about economies with production

Robinson Crusoe

Two firms

General Economies with Many Consumers and Production

Lecture 6: General Equilibrium

A few things I forgot to say about economies with production

Robinson Crusoe

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General Economies with Many Consumers and Production

- With production, Edgeworth box illustrations are no longer helpful
- Depending on the production plan, the size of the box can change
- Instead we work with what is called a production possibilities frontier

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General Economies with Many Consumers and Production

- Imagine the problem of Robinson Crusoe, living alone in an island. He is the only producer and the only consumer

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Suppose that the consumer (Robinson) has a utility function:

$$u(L, x),$$

where x are coconuts. There is one firm (Robinson) that can convert labor to coconuts:

$$f(L_x)$$

The endowment is $(0, \bar{L})$

What is the Pareto optimal allocation in this economy?

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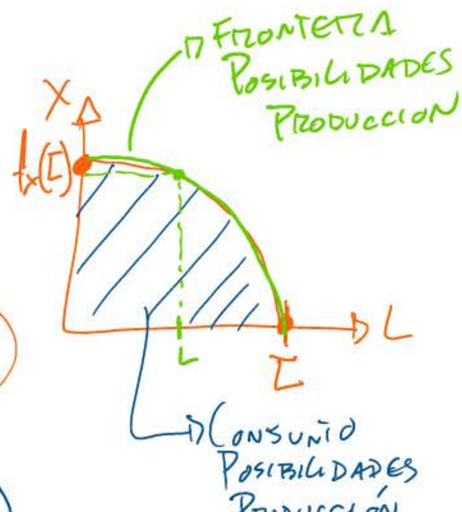
$$\max u(L, x) \text{ ~~such that~~ } x \le f(L_x)$$
$$L_x + L \le \bar{L}.$$

FACTIBLE.

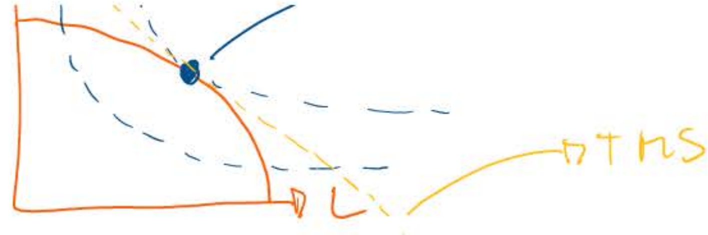
$\bar{L} = L_x + L$

$x \le f(\underbrace{\bar{L} - L}_{L_x})$



Or equivalently

$$\max u(L, f(\bar{L} - L))$$



TMT = TMS

Or equivalently

$$\max u(L, f(\bar{L} - L))$$

We can solve this either using calculus or graphically

Using calculus...

$$\max U(x, L) \text{ s.a } x = f_x(\bar{L} - L)$$

$$\max U(f_x(\bar{L} - L), L)$$

$$\frac{\partial U(f_x(\bar{L}-L), L)}{\partial L} = \frac{\partial U}{\partial x} \cdot \frac{\partial f_x}{\partial L_x} \cdot (-1) + \frac{\partial U}{\partial L} = 0$$

$$\boxed{TMS = \frac{\frac{\partial U}{\partial L}}{\frac{\partial U}{\partial x}} = \frac{\partial f_x}{\partial L_x} = TMT}$$

Using calculus... This is the order condition:

$$\frac{\partial u}{\partial L}(L, f(\bar{L} - L)) - \frac{\partial u}{\partial x}(L, f(\bar{L} - L)))f'(\bar{L} - L) = 0$$

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$$\frac{\partial u}{\partial L}(L, f(\bar{L} - L)) = \frac{\partial u}{\partial x}(L, f(\bar{L} - L)))f'(\bar{L} - L)$$

$$\frac{\frac{\partial u}{\partial L}(L, f(\bar{L} - L))}{\frac{\partial u}{\partial x}(L, f(\bar{L} - L)))} = f'(\bar{L} - L)$$

$$f'(\bar{L} - L) = \frac{\frac{\partial u}{\partial L}(L, f(\bar{L} - L))}{\frac{\partial u}{\partial x}(L, f(\bar{L} - L))} = MRS_{L,x}$$

- If Robinson gives up 1 unit of consumption in L, $f'(\bar{L}-L)$ describes how much more in terms of x Robinson will be able to consume

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- If Robinson gives up 1 unit of consumption in L, $f'(\bar{L}-L)$ describes how much more in terms of x Robinson will be able to consume
- This is what is called a **Marginal Rate of Transformation** of good L to x

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$$f'(\bar{L}-L) = \frac{\frac{\partial u}{\partial L}(L, f(\bar{L}-L))}{\frac{\partial u}{\partial x}(L, f(\bar{L}-L))} = MRS_{L,x}$$

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$$MRT_{L,x} = MRS_{L,x}.$$

We can solve the above graphically.

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$$\{(L,x) : x \leq f(L_X), L_x + L \leq \bar{L}\}$$

describes the possible sets of bundles that Robinson could possibly consume in this economy.

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This is called the **production possibilities set** (PPS)

We can solve the above graphically. The set

$$\{(L, x) : x \leq f(L_X), L_x + L \leq \bar{L}\}$$

describes the possible sets of bundles that Robinson could possibly consume in this economy.
This is called the **production possibilities set** (PPS)
The boundary of the PPS is the**production possibilities frontier** (PPF)

The frontier is basically described by the curve:

$$x = f(L_x), L_x \in [0, \bar{L}].$$

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$$x = f(L_x), L_x \in [0, \bar{L}].$$

The maximization problem for finding Pareto efficient allocations simply amounts to maximizing the utility of Robinson subject to being inside this constraint set.

Lecture 6: General Equilibrium

- Recall that at a Pareto optimum, we found that we must have $MRT_{L,x} = MRS_{L,x}$
- Suppose one is at an allocation where $MRT_{L,x} = 2 > MRS_{L,x} = 1$
- Such an allocation cannot be a Pareto efficient allocation. Why?

- Recall that at a Pareto optimum, we found that we must have $MRT_{L,x} = MRS_{L,x}$
- Suppose one is at an allocation where $MRT_{L,x} = 2 > MRS_{L,x} = 1$
- Such an allocation cannot be a Pareto efficient allocation. Why?
- One could potentially reorganize production to get an even better outcome for the consumer

A few things I forgot to say about economies with production

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General Economies with Many Consumers and Production

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General Economies with Many Consumers and Production

- The consumer has a utility function $u(x, y)$
- The consumer is endowed with 0 units of both x and y but x and y can be produced from labor and capital
- She is endowed with K units of capital and L units of labor
- There are two firms each of which produces a commodity x and y.
- Firm x produces x according to a production function f_x and firm y produces y according to a production function f_y:

$$f_x(\ell_x, k_x), f_y(\ell_y, k_y).$$

To solve for the Pareto efficient allocation we solve:

$$\max u(x, y) \text{ such that } \boxed{\begin{array}{l} x \leq f_x(\ell_x, k_x), y \leq f_y(\ell_y, k_y), \\ L \geq \ell_x + \ell_y, K \geq k_x + k_y. \end{array}}$$

FACTIBLE.

A few things I forgot to say about economies with production

Robinson Crusoe
- Econ 1 Intuition
- A concrete example

Two firms
- Graphical Approach
- Calculus Approach I
- Calculus Approach II
- Concrete Example

General Economies with Many Consumers and Production
- Solving the Maximization Problem

We look at the production possibilities set (PPS) of this economy:

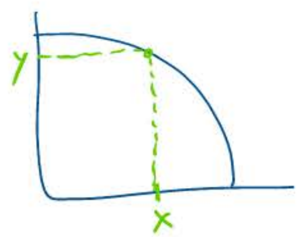
$$\{(x,y) : x \le f_x(\ell_x,k_x), y \le f_y(\ell_y,k_y), \ell_x+\ell_y \le L, k_x+k_y \le K\}.$$

TMS = TMT

Then given the PPS, we want to maximize the utility of the agent subject to being inside the PPS. If we want to maximize the utility of the agent, we need:

1. The chosen (x^*,y^*) must be on the PPF.

2. The indifference curve of the consumer must be tangent to the PPF at (x^*,y^*).



Lecture 6: General Equilibrium

To find the PPF: given that x units of commodity x must be produced, what is the maximum amount of y's that can be produced?

To find the PPF: given that x units of commodity x must be produced, what is the maximum amount of y's that can be produced? Thus

$$PPF(x) = \boxed{\max_{\ell_y,k_y} f_y(\ell_y,k_y)} \text{ such that } \bar x = f_x(\overbrace{L-\ell_y}^{\ell_x}, \overbrace{K-k_y}^{k_x}).$$

$$\mathcal{L} = f_y(L_y,K_y) + \lambda\left(f_x(L-L_y,K-K_y) - \bar x\right)$$

$$\frac{\partial \mathcal{L}}{\partial L_y} = \frac{\partial f_y}{\partial L_y} + \lambda \frac{\partial f_x}{\partial L_x}(-1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_y} = \frac{\partial f_y}{\partial K_y} + \lambda \frac{\partial f_x}{\partial K_x}(-1) = 0$$

$$\Rightarrow \quad \frac{\frac{\partial f_y}{\partial L_y}}{\frac{\partial f_y}{\partial K_y}} = \frac{\frac{\partial f_x}{\partial L_x}}{\frac{\partial f_x}{\partial K_x}} \Rightarrow \boxed{\frac{\text{UNIDADES } K}{\text{UNIDADES } L}}$$

$TMST_x = TMST_y$

$2 = TMST_x > TMST_y = 1$

Setting up the Lagrangian we get:

$$\max_{\ell_y,k_y} f_y(\ell_y,k_y) + \lambda(f_x(L-\ell_y, K-k_y) - x))$$

The first order conditions give us:

$$\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*) - \lambda \frac{\partial f_x}{\partial \ell}(L - \ell_y^*, K - k_y^*) = 0$$
$$\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) - \lambda \frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*) = 0$$

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$$\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) - \lambda \frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*) = 0$$

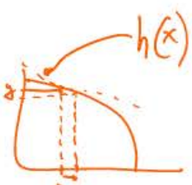
$$\boxed{\begin{aligned} \frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*) &= \lambda \frac{\partial f_x}{\partial \ell}(L - \ell_y^*, K - k_y^*) \\ \frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*) &= \lambda \frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*) \end{aligned}}$$

Thus at the optimum, we have:

$$TRS^y_{\ell,k} = \frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)} = \frac{\frac{\partial f_x}{\partial \ell}(L - \ell_y^*, K - k_y^*)}{\frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*)} = TRS^x_{\ell,k}.$$

- To actually solve for the optimal x^* and y^* we plug this back into the constraint $x = f_x(L - \ell_y^*, K - k_y^*)$
- Therefore at a Pareto optimum we must have $TRS^y_{\ell,k} = TRS^x_{\ell,k}$ equalized
- You should be able to come up with the Econ 1 intuition for this as we have done previously

- A Pareto optimum also requires bullet point 2 above (i.e., indifference curve is tangent to PPF)
- The slope of the indifference curve is given by:
$$\boxed{-MRS_{x,y} = -\frac{\frac{\partial u}{\partial x}(x^*, y^*)}{\frac{\partial u}{\partial y}(x^*, y^*)}.}$$
- What is the slope of the PPF?



- Note that mathematically, this is given by $PPF'(x)$
- How do we calculate that?
$$\boxed{PPF(\bar{x}) = \max_{\ell_y, k_y} f_y(\ell_y, k_y) + \lambda(f_x(L - \ell_y, K - k_y) - \bar{x}))}$$
- By the envelope theorem
$$\begin{aligned} PPF'(x) &= -\lambda \\ &= -\frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial \ell}(L - \ell_y^*, K - k_y^*)} \\ &= -\frac{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*)} \\ &= -MRT_{x,y} \end{aligned}$$

$$\frac{\text{Unidades } Y}{\text{Unidades } X} = TMT$$

- Note that mathematically, this is given by $PPF'(x)$
- How do we calculate that?

$$PPF(x) = \max_{\ell_y, k_y} f_y(\ell_y, k_y) + \lambda(f_x(L - \ell_y, K - k_y) - x))$$

- By the envelope theorem

$$\begin{aligned} PPF'(x) &= -\lambda \\ &= -\frac{\frac{\partial f_y}{\partial \ell}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial \ell}(L - \ell_y^*, K - k_y^*)} \\ &= -\frac{\frac{\partial f_y}{\partial k}(\ell_y^*, k_y^*)}{\frac{\partial f_x}{\partial k}(L - \ell_y^*, K - k_y^*)} \\ &= -MRT_{x,y} \end{aligned}$$

- Therefore, at a Pareto optimum:

$$\boxed{MRT_{x,y} = MRS_{x,y}.}$$

Thus we have learned the following: A Pareto efficient allocation is characterized by two conditions:

1. (x^*, y^*) is on the PPF: $TRS^x_{\ell,k} = TRS^y_{\ell,k}$.
2. At (x^*, y^*) the indifference curve is tangent to the PPF: $MRS_{x,y} = MRT_{x,y}$.

Lecture 6: General Equilibrium

We solve directly the original maximization problem.

$$\begin{aligned} \max u(x, y) \text{ such that } & x \leq f_x(\ell_x, k_x), y \leq f_y(\ell_y, k_y), \\ & L \geq \ell_x + \ell_y, K \geq k_x + k_y. \end{aligned}$$

We can simplify the problem:

$$\max_{\ell_x, k_x} u(f_x(\ell_x, k_x), f_y(L - \ell_x, K - k_x))$$

Then the first order conditions give us:

$$\begin{aligned} &\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L - \ell_x^*, K - k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L - \ell_x^*, K - k_x^*))\frac{\partial f_y}{\partial \ell}(L - \ell_x^*, K - k_x^*) = 0, \\ &\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L - \ell_x^*, K - k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L - \ell_x^*, K - k_x^*))\frac{\partial f_y}{\partial k}(L - \ell_x^*, K - k_x^*) = 0 \end{aligned}$$

Then the first order conditions give us:

$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*) = 0,$$
$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*, k_x^*) - \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial k}(L-\ell_x^*, K-k_x^*) = 0$$

$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*) = \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*),$$
$$\frac{\partial u}{\partial x}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_x}{\partial k}(\ell_x^*, k_x^*) = \frac{\partial u}{\partial y}(f_x(\ell_x^*, k_x^*), f_y(L-\ell_x^*, K-k_x^*))\frac{\partial f_y}{\partial k}(L-\ell_x^*, K-k_x^*).$$

We obtain:

$$TRS^x_{\ell,k} = TRS^y_{\ell,k},$$
$$MRS_{x,y} = MRT_{x,y}.$$

Lecture 6: General Equilibrium

Suppose that the utility function are given by:

$$u(x, y) = \sqrt{xy}$$

and suppose that the production functions are given by:

$$f_x(\ell_x, k_x) = \sqrt{\ell_x k_x}, f_y(\ell_y, k_y) = \sqrt{\ell_y k_y}.$$

Then Pareto efficiency involves solving the following maximization problem:

$$\max \sqrt{xy} \text{ such that } x = f_x(\ell_x, k_x), y = f_y = f_y(\ell_y, k_y).$$

$\ell_x + \ell_y = \bar{L}$
$k_x + k_y = \bar{K}$

FACTIBLE

Approach 1: First lets characterize the PPF.

$$PPF(x) = \boxed{\max f_y(L-\ell_x, K-k_x)} \text{ such that } f_x(\ell_x, k_x) = x.$$

$= \max (L-\ell_x)^{1/2}(K-k_x)^{1/2}$ s.a $L_x^{1/2} k_x^{1/2} = x$

$\frac{\partial y}{\partial L_x} = \frac{1}{2}(L-L_x)^{-1/2}(-1)(K-k_x)^{1/2} - \lambda\left(\frac{1}{2}L_x^{-1/2}k_x^{1/2}\right) = 0$

$\frac{\partial y}{\partial k_x} = \frac{1}{2}(L-L_x)^{1/2}(K-k_x)^{-1/2}(-1) - \lambda\left(\frac{1}{2}L_x^{1/2}k_x^{-1/2}\right) = 0$

$\Rightarrow \frac{(K-k_x)}{L-L_x} = \frac{k_x}{L_x}$

$K L_x - \cancel{k_x L_x} = k_x L - \cancel{k_x L_x}$

$\boxed{L_x = k_x \frac{L}{K}}$

By the first order condition, we need:

$$\frac{k_x^*}{\ell_x^*} = \frac{\frac{\partial f_x}{\partial \ell}(\ell_x^*, k_x^*)}{\frac{\partial f_x}{\partial k}(\ell_x^*, k_x^*)} = \frac{\frac{\partial f_y}{\partial \ell}(L-\ell_x^*, K-k_x^*)}{\frac{\partial f_y}{\partial k}(L-\ell_x^*, K-k_x^*)} = \frac{K-k_x^*}{L-\ell_x^*} \Longrightarrow k_x^* = \frac{K}{L}\ell_x^*.$$

Plug this back into the constraint:

$$f_x(\ell_x^*, k_x^*) = x \Longrightarrow \sqrt{\ell_x k_x} = x \Longrightarrow \ell_x^* = \sqrt{\frac{L}{K}}x.$$

Therefore

$$f_y(L-\ell_x, K-k_x) = y$$

$$\sqrt{\left(L-\sqrt{\frac{L}{K}}x\right)\left(K-\sqrt{\frac{K}{L}}x\right)} = y$$

$$PPF(x) = \left(\sqrt{KL}-x\right)$$

$K L_x - \cancel{K_x L_x} = K_x L - \cancel{K_x L_x}$

$\boxed{L_x = K_x \frac{L}{K}}$

$L_x^{1/2} K_x^{1/2} = X$

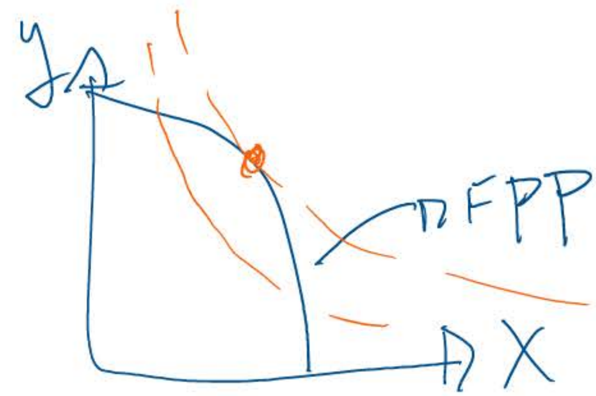
$K_x^{1/2}\left(\frac{L}{K}\right)^{1/2} K_x^{1/2} = X$

$\boxed{K_x = X\left(\frac{K}{L}\right)^{1/2}}$

$\boxed{L_x = X\left(\frac{L}{K}\right)^{1/2}}$

$y^* = (L-L_x)^{1/2}(K-K_x)^{1/2}$

$y^* = \left(L - X\left(\frac{L}{K}\right)^{1/2}\right)^{1/2}\left(K - X\left(\frac{K}{L}\right)^{1/2}\right)^{1/2}$



Then we need to maximize the following:

$$\max_{x,y} \sqrt{xy} \text{ such that } \boxed{y = \sqrt{KL} - x}.$$

FPP

$\mathcal{L} = \sqrt{xy} + \lambda(y - \sqrt{KL} + x)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}x^{-1/2}y^{1/2} + \lambda = 0 \Rightarrow \frac{y}{x} = 1$

$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}x^{1/2}y^{-1/2} + \lambda = 0$

$\boxed{y = x}$

$y = \sqrt{KL} - x$

$2x = \sqrt{KL}$

$\boxed{x = \frac{1}{2}\sqrt{KL}}$

The Pareto efficient allocation is given by:

$$\left(x^* = y^* = \frac{1}{2}\sqrt{KL}, \ell_x^* = \ell_y^* = \frac{1}{2}L, k_x^* = k_y^* = \frac{1}{2}K\right).$$

Lecture 6: General Equilibrium

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A few things I forgot to say about economies with production

Robinson Crusoe

Two firms

General Economies with Many Consumers and Production

The set of Pareto efficient allocations will be characterized by the following maximization problem:

$$\max_{(x,z)} u_1(x_1^1, x_2^1, \dots, x_L^1) \text{ such that } u_2(x_1^2, \dots, x_L^2) \geq \underline{u}_2 = u_2(\hat{x}_1^2, \dots, \hat{x}_L^2),$$

$$\vdots$$

$$u_I(x_1^I, \dots, x_L^I) \geq \underline{u}_I = u_I(\hat{x}_1^I, \dots, \hat{x}_L^I),$$

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$$x_1^1 + \cdots x_1^I + z_1^1 + \cdots + z_1^J \leq \sum_{j: \ell(j)=1} f_j(z_1^j, \dots, z_L^j) + \sum_{i=1}^{I} \omega_1^i,$$

$$\vdots$$

$$x_L^1 + \cdots x_L^I + z_L^1 + \cdots + z_L^J \leq \sum_{j: \ell(j)=L} f_j(z_1^j, \dots, z_L^j) + \sum_{i=1}^{I} \omega_L^i.$$

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Lecture 6: General Equilibrium

A few things I forgot to say about economies with production

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- Econ 1 Intuition
- A concrete example

Two firms
- Graphical Approach
- Calculus Approach I
- Calculus Approach II
- Concrete Example

General Economies with Many Consumers and Production
- Solving the Maximization Problem

Theorem

Suppose that utility functions are strictly monotone, differentiable, and quasi-concave. Suppose also that $(\hat{x}, \hat{z})$ is an interior allocation. Then $(\hat{x}, \hat{z})$ is Pareto efficient if and only if all of the following hold:

1. *For every $\ell \neq \ell'$, marginal rates of substitution of any pair of commodities are equalized across consumers:*

$$\frac{\frac{\partial u_1}{\partial x_\ell}(\hat{x}_1^1, \dots, \hat{x}_L^1)}{\frac{\partial u_1}{\partial x_{\ell'}}(\hat{x}_1^1, \dots, \hat{x}_L^1)} = \frac{\frac{\partial u_2}{\partial x_\ell}(\hat{x}_1^2, \dots, \hat{x}_L^2)}{\frac{\partial u_2}{\partial x_{\ell'}}(\hat{x}_1^2, \dots, \hat{x}_L^2)} = \cdots = \frac{\frac{\partial u_I}{\partial x_\ell}(\hat{x}_1^I, \dots, \hat{x}_L^I)}{\frac{\partial u_I}{\partial x_{\ell'}}(\hat{x}_1^I, \dots, \hat{x}_L^I)}.$$

2. *For every $\ell \neq \ell'$, technical rates of substitution of inputs ℓ and ℓ' are equalized across firms:*

$$\frac{\frac{\partial f_1}{\partial z_\ell}(\hat{z}_1^1, \dots, \hat{z}_L^1)}{\frac{\partial f_1}{\partial z_{\ell'}}(\hat{z}_1^1, \dots, \hat{z}_L^1)} = \frac{\frac{\partial f_2}{\partial z_\ell}(\hat{z}_1^2, \dots, \hat{z}_L^2)}{\frac{\partial f_2}{\partial z_{\ell'}}(\hat{z}_1^2, \dots, \hat{z}_L^2)} = \cdots = \frac{\frac{\partial f_J}{\partial z_\ell}(\hat{z}_1^J, \dots, \hat{z}_L^J)}{\frac{\partial f_J}{\partial z_{\ell'}}(\hat{z}_1^J, \dots, \hat{z}_L^J)}.$$

3. *For every $\ell \neq \ell'$, the marginal rates of transformation is equal to the marginal rates of substitution:*

$$\frac{\frac{\partial f_j}{\partial z_{\ell''}}(\hat{z}_1^j, \dots, \hat{z}_L^j)}{\frac{\partial f_{j'}}{\partial z_{\ell''}}(\hat{z}_1^{j'}, \dots, \hat{z}_L^{j'})} = \frac{\frac{\partial u_1}{\partial x_\ell}(\hat{x}_1^1, \dots, \hat{x}_L^1)}{\frac{\partial u_1}{\partial x_{\ell'}}(\hat{x}_1^1, \dots, \hat{x}_L^1)} = \cdots = \frac{\frac{\partial u_I}{\partial x_\ell}(\hat{x}_1^I, \dots, \hat{x}_L^I)}{\frac{\partial u_I}{\partial x_{\ell'}}(\hat{x}_1^I, \dots, \hat{x}_L^I)}.$$