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Lecture12....

Lecture 12: Game Theory // Nash equilibrium

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Lecture 12: Game Theory // Nash equilibrium

Dominance

Nash equilibrium

Some examples

Relationship to dominance

Examples

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Relationship to dominance

Examples

Beauty contest

- Consider the following game among 100 people. Each individual selects a number, s_i, between 20 and 60.
- Let a_{-i} be the average of the number selected by the other 99 people. i.e. $a_{-i} = \sum_{j \neq i} \frac{s_j}{99}$.
- The utility function of the individual i is $\boxed{u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}a_{-i})^2}$

Beauty contest

- Each individual maximizes his utility, FOC:

$$-2(s_i - \frac{3}{2}a_{-i}) = 0$$

$s_i^* = \frac{3}{2}\underline{a_{-i}}$

$a_{-i} \in [20, 60]$

$\frac{3}{2}a_{-i} \in [30, 90]$

Por eliminación estrategias dominadas

$s_i \in [30, 60]$

$a_{-i} \in [30, 60]$

$\frac{3}{2}a_{-i} \in [45, 90]$

Por eliminación

$s_i \in [45, 60]$

$a_{-i} \in [45, 60]$

$\frac{3}{2}a_{-i} \in [67.5, 90]$

$\Rightarrow$ Por eliminación

$s_i = 60$

Beauty contest

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- but $a_{-i} \in [20, 60]$

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- Individuals would prefer to select a number that is exactly equal to 1.5 times the average of the others
- That is they would like to choose $s_i = \frac{3}{2}a_{-i}$
- but $a_{-i} \in [20, 60]$
- Therefore $s_i = 20$ is dominated by $s_i = 30$

Beauty contest

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- Playing a number between 30 and 45 (not including) would be strictly dominated by playing 45

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- Knowing this, all individuals believe that everyone else will select a number between 45 and 60 (i.e., $a_{-i} \in [45, 60]$)
- 60 would dominate any other selection and therefore all the players select 60.
- The solution by means of iterated elimination of dominated strategies is $\underbrace{(60, 60, \ldots, 60)}_{\text{100 times}}$

Lecture 12: Game Theory // Nash equilibrium

Dominance
- Weakly dominated strategies

Nash equilibrium

Some examples

Relationship to dominance

Examples
- Cournot Competition
- Cartels

	a	b
~~A~~	~~3, 4~~	~~4, 3~~
~~B~~	~~5, 3~~	~~3, 5~~
C	5, 3	~~4, 3~~

$C > A$ ↳ Debil

$C > B$ ↳ Debil

- There is no strictly dominated strategy

$C > A \to b > a \to C > B \to (C, b)$

$C > A \to C > B \to (C, a)$ y (C, b)

$C > B \to a > b \to C > A \to (C, a)$

	a	b
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- There is no strictly dominated strategy
- However, C always gives at least the same utility to player 1 as B
- It's tempting to think player 1 would never play C
- However, if player 1 is sure that player two is going to play a he would be completely indifferent between playing B or C

Definition
s_i weakly dominates s_i' if for all opponent pure strategy profiles, $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

and there is at least one opponent strategy profile $s''_{-i} \in S_{-i}$ for which

$$u_i(s_i, s''_{-i}) > u_i(s_i', s''_{-i}).$$

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- Rationality is not enough
- Even so, it sounds "logical" to do so and has the potential to greatly simplify a game
- There is a problem, and that is that the order in which we eliminate the strategies matters

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
C	5, 3	4, 3

- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).

	a	b
A	3, 4	4, 3
B	5, 3	3, 5
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- If we eliminate B (C dominates weakly), then a weakly dominates b and we can eliminate b and therefore player 1 would never play A. This leads to the result (C, a).
- If on the other hand, we notice that A is also weakly dominated by C then we can eliminate it in the first round, and this would eliminate a in the second round and therefore B would be eliminated. This would result in (C, b).

Lecture 12: Game Theory // Nash equilibrium

Lecture 12: Game Theory // Nash equilibrium

Remember the definition of competitive equilibrium in a market economy.

Definition
A competitive equilibrium in a market economy is a vector of prices and baskets x_i such that: 1) x_i maximizes the utility of each individual given the price vector i.e.

$$x_i = \arg \max_{p \cdot x_i \leq p \cdot w_i} u(x_i)$$

2) the markets empty.

$$\sum_i x_i = \sum_i w_i$$

- 1) means that given the prices, individuals have no incentive to demand a different amount

- 1) means that given the prices, individuals have no incentive to demand a different amount
- The idea is to extend this concept to strategic situations

Best response

We denote $BR_i(s_{-i})$ (best response) as the set of strategies of individual i that maximize her utility given that other individuals follow the strategy profile s_{-i}. Formally,

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Given a strategy profile of opponents s_{-i}, we can define the best response of player i:

$$BR_i(s_{-i}) = \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}).$$

- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$

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- $s_i \in BR_i(s_{-i})$ if and only if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$
- There could be multiple strategies in $BR_i(s_{-i})$ but all such strategies give the same utility to player i if the opponents are indeed playing according to s_{-i}

Nash equilibrium

Definition
Suppose that we have a game $(I = \{1, 2, \ldots n\}, S_1, \ldots, S_n, u_1, \ldots, u_n)$. Then a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a pure strategy **Nash equilibrium** if for every i and for every $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

$\forall i \quad s_i^* \in BR_i(S^*_{-i})$

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- Analogous to that of a competitive equilibrium in the sense that nobody has unilateral incentives to deviate
- once this equilibrium is reached, nobody has incentives to move from there
- This is a concept of stability, but there is no way to ensure, or predict, that the game will reach this equilibrium

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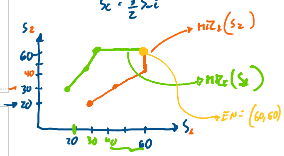
$u_i(s_i, s_{-i}) = 100 - (s_i - \frac{3}{2}s_{-i})^2$

$\frac{\partial u_i}{\partial s_i} = -2(s_i - \frac{3}{2}s_{-i}) = 0$

$s_i^* = \frac{3}{2}s_{-i}$

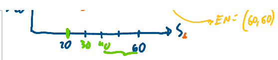


Beauty contest

The best response of an individual is given by

$$s_i(s_{-i})^* = \begin{cases} \frac{3}{2}s_{-i} & \text{if } s_{-i} \leq 40 \\ 60 & \text{if } s_{-i} > 40 \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)



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The Nash equilibrium is where both BR functions intersect (i.e., when both play 60)

Prisoner's dilemma

S_2

S_1

	C	NC
C	5,5	0,10
NC	10,0	2,2

$MR_1(S_2=C)=NC$
$MR_1(S_2=NC)=NC$

$MR_2(S_1=C)=NC$
$MR_2(S_1=NC)=NC$

$\boxed{EN=(NC,NC)}$

Prisoner's dilemma

	C	NC
C	5,5	0,10
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The best response functions are:

$$BR_i(s_{-i}) = \begin{cases} NC & \text{if } s_{-i} = C \\ NC & \text{if } s_{-i} = NC \end{cases}$$

The Nash equilibrium is where both BR functions intersect (i.e., when both play NC, i.e., (NC, NC))

Prisoner's dilemma – A trick

Best response of 1 to 2 playing C

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 1 to 2 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing C

	C	NC
C	5,5	0,10
NC	10,0	2,2

Prisoner's dilemma – A trick

Best response of 2 to 1 playing NC

	C	NC
C	5,5	0,10
NC	10,0	2,2

When underlined for both players, it is a Nash equilibrium (both are doing their BR)

Battle of the sexes

	G	P
G	2,1	0,0
P	0,0	1,2

$EN = \{(G,G), (P,P)\}$.

Battle of the sexes

	G	P
G	2,1	0,0
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$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Battle of the sexes

	G	P
G	2,1	0,0
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$$BR_i(s_{-i}) = \begin{cases} G & \text{if } s_{-i} = G \\ P & \text{if } s_{-i} = P \end{cases}$$

Thus, (G, G) y (P, P) are both Nash equilibrium

Matching pennies (Pares o Nones) – Simultaneous

	1	2
1	(1000,-1000)	(-1000,1000)
2	(-1000,1000)	(1000,-1000)

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$$BR_1(s_2) = \begin{cases} 1 & \text{if } s_2 = 1 \\ 2 & \text{if } s_2 = 2 \end{cases}$$

$$BR_2(s_1) = \begin{cases} 2 & \text{if } s_1 = 1 \\ 1 & \text{if } s_2 = 2 \end{cases}$$

There is no Nash equilibrium in pure strategies

Lecture 12: Game Theory // Nash equilibrium

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Nash equilibrium survive IDSDS

Theorem
Every Nash equilibrium survives the iterative elimination of strictly dominated strategies

Proof

By contradiction:

- Suppose it is not true

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- Lets zoom in in the round where we first eliminate a strategy that is part of s^*
- Without loss of generality say we eliminated the strategy s_i^* of individual i
- It must have been that

$$u_i(s_i^*, s_{-i}) < u_i(s_i, s_{-i}) \forall \underline{s_{-i} \in S_{-i}}$$

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$$u_i(s_i^*, s_{-i}^*) < u_i(s_i, s_{-i}^*)$$

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- But this means s_i^* is not the best response of individual i to s_{-i}^*

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- But this means s_i^* is not the best response of individual i to s_{-i}^*
- And this is a contradiction!

Nash equilibrium survive IDSDS

Theorem
If the process of IDSDS comes to a single solution, that solution is a Nash Equilibrium and is unique.

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

Proof.
By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium

□

Proof

First let's proof its a Nash Equilibrium. The fact that is unique is trivial by the previous theorem.

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By contradiction:

- Suppose that the results from IDSDS (s^*) is not a Nash Equilibrium
- For some individual i there exits s_i such that

$$u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$

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- But then s_i could not have been eliminated

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Cournot Competition

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$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

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- If firm 1 and 2 produce q_1 and q_2 units of the commodity respectively, the inverse demand function is given by:

$$P(Q) = 120 - Q, Q = q_1 + q_2.$$

- Strategy space is $S_i = [0, +\infty)$
- The utility function of player i is given by:

$$\pi_1(q_1, q_2) = (120 - (q_1 + q_2))q_1 - c(q_1) \to 0$$
$$\pi_2(q_1, q_2) = (120 - (q_1 + q_2))q_2 - c(q_2)$$

$\pi_1(40,40) = (120 - 80)40$

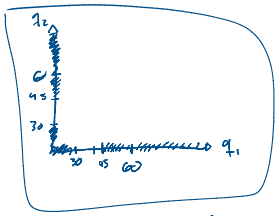
Cournot Competition

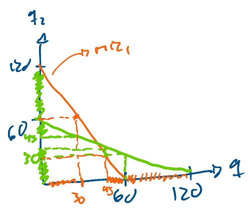
- Are there any strictly dominant strategies?

$\frac{\partial \pi_1}{\partial q_1} = 120 - 2q_1 - q_2 = 0$ $\quad \boxed{\frac{120 - q_2}{2} = q_1}$

$\frac{\partial \pi_2}{\partial q_2} = 120 - q_1 - 2q_2 = 0$

$\frac{120 - q_1}{2} = q_2 = MR_2$





Cournot Competition

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$\boxed{MR_1(q_2) = \frac{120 - q_2}{2} = q_1}$

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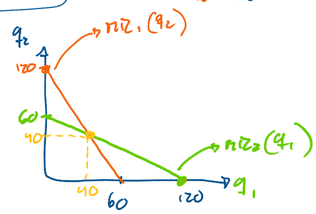

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?

$120 - q_2 = 2q_1$

$120 - q_1 = 2q_2 \to 120 - 2q_2 = q_1$

$120 - q_2 = 2(120 - 2q_2)$

$120 - q_2 = 240 - 4q_2$

$3q_2 = 120$

$\boxed{q_2 = 40}$ $\quad \boxed{q_1 = 40}$

$EN = (40, 40)$

$GANANCIA_{EN} = (40^2, 40^2) = (1600, 1600)$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0

$$\max_{q_1, q_2} \Pi = \Pi_1 + \Pi_2 = (120 - (q_1 + q_2))q_1 + (120 - (q_1 + q_2))q_2$$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$

$$\frac{\partial \Pi}{\partial q_1} = 120 - 2q_1 - q_2 - q_2 = 0$$

$$\frac{\partial \Pi}{\partial q_2} = -q_1 + 120 - q_1 - 2q_2 = 0$$

$\Pi^m = (120 - q)q$

$\frac{\partial \Pi^m}{\partial q} = 120 - 2q^m = 0$

$\boxed{q^m = 60}$

Cournot Competition

- Are there any strictly dominant strategies? The answer is no, why?
- Are there any strictly dominated strategies?
- The strategies $q_i \in (120, +\infty)$ are strictly dominated by the strategy 0
- Are there any others? given q_{-i},

$$\frac{d\pi_i}{dq_i}(120 - q_i - q_{-i})q_i = 120 - 2q_i - q_{-i}$$
- Therefore 60 strictly dominates any $q_i \in (60, 120]$

$$120 - 2q_1 - 2q_2 = 0$$

$$\boxed{60 = \frac{120}{2} = q_1 + q_2}$$

Cournot Competition

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Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- for any $q_i \in [0, 60]$, there exists some $q_{-i} \in [0, +\infty)$ such that $BR_i(q_{-i}) = q_i$

CARTEL

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- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [0, 60]$
- Therefore $q_i \in [0, 30)$ are strictly dominated by $q_i = 30$
- After two rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 60]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 60]$
- 45 strictly dominates all strategies $q_i \in (45, 60]$
- After three rounds of deletion of strictly dominated strategies, we are left with: $S_i = [30, 45]$

Cournot Competition

- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$
- $q_{-i} = [30, 45]$
- 37.5 strictly dominates all strategies $q_i \in [30, 37.5]$
- After four rounds of deletion of strictly dominated strategies, we are left with: $S_i = [37.5, 45]$

Cournot Competition

- After (infinitely) many iterations, the only remaining strategies are $S_i = 40$
- The unique solution by IDSDS is $q_1^* = q_2^* = 40$.

Cournot Competition

- There will also be a unique Nash equilibrium

Cournot Competition

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- $$BR_i(q_{-i}) = \frac{120 - q_{-i}}{2}.$$

$$q_1 = q_2 = 30$$

$$\pi_1(30,30) = (120 - (30+30))30$$
$$= 60 \cdot 30 = 1800$$

$$\pi_1(30,30) = \pi_2(30,30) = 1,800$$

$$BR_1(30) = \frac{120-30}{2} = 45$$

OPEC

Cournot Competition

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- $$q_1^* = \frac{120 - q_2^*}{2}, q_2^* = \frac{120 - q_1^*}{2}.$$
- We can solve for q_1^* and q_2^* to obtain:
$$q_1^* = 40, q_2^* = 40, Q^* = 80, \Pi_1^* = \Pi_2^* = 1600.$$

Cournot Competition vs Monopoly (cartel)

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- In a duopoly, externalities are imposed on the other firm

Lecture 12: Game Theory // Nash equilibrium

Cartels

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- The inverse demand function is given by:
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$$1 - 2q_i - Q_{-i} = 0 \Longrightarrow q_i = \frac{1 - Q_{-i}}{2} \Longrightarrow BR_i(Q_{-i}) = \frac{1 - Q_{-i}}{2}.$$
- In a Nash equilibrium we must have:
$$q_1^* = \frac{1 - q_2^* - q_3^*}{2}$$
$$q_2^* = \frac{1 - q_1^* - q_3^*}{2}$$
$$q_3^* = \frac{1 - q_1^* - q_2^*}{2}.$$

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- $q_1^* = q_2^* = q_3^* = \frac{1}{4}$
- Price is $p^* = 1/4$ and all firms get the same profits of $1/16$

Cartels

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- Therefore

$$q_A^* = \frac{1 - q_B^*}{2}$$
$$q_B^* = \frac{1 - q_A^*}{2}.$$

Cartels

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- Firms 1 and 2 suffered, while firm 3 is better off!
- Firm 3 is obtaining a disproportionate share of the joint profits (more than $1/3$)

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- Total profits then are given by $\frac{1}{4}$ which means that each firm obtains a profit of $\frac{1}{12} < \frac{1}{9}$
- Firm 3 clearly wants to stay out

Cartels

There are many difficulties associated with sustaining collusive agreements (e.g., the OPEC cartel)