# Lecture13

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Lecture13

Lecture 13: Game Theory // Nash equilibrium

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Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

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Examples - Continued

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued
- Cournot - Revisited
- Bertrand Competition
- Bertrand Competition - Different costs
- Bertrand Competition - 3 Firms
- Hotelling and Voting Models

## Cournot Competition

- $N$ identical firms competing on the same market

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- $N$ identical firms competing on the same market
- Marginal cost is constant and equal to $c$
- Aggregate inverse demand is

$$p = a - b\sum_{j=1}^{N} q^j$$

- Benefits of firm $j$ are:

$$\Pi^j(q^1, \dots q^N) = \left(a - b\sum_{i=1}^{N} q^i\right) q^j - cq^j.$$

$$\Pi(q^*, \dots, q^*) = (a - bNq^*)q^* - cq^*$$

$$\frac{\partial \Pi}{\partial q^*} = a - 2bNq^* - c = 0$$

$$\frac{a-c}{2bN} = q^*$$

"j"

$$\frac{\partial \Pi^j}{\partial q_j} = -bq^j + \left(a - b\sum q_i\right)\cdot 1 - c = 0$$

$$-2bq^j + a \underbrace{- bQ_{-j}}_{\sum_{i \neq j} q_i} - c = 0 \longrightarrow q^j = mr^j(Q_{-j}) = \frac{a - c - bQ_{-j}}{2b}$$

$$a - b(q_1 + q_2 + \dots + 2q_j + \dots + q_N) - c = 0$$

Vamos a buscar un E.N. simetrico

$$q_1^* = q_2^* = \dots = q_N^* = q^*$$

$$a - b(N+1)q^* - c = 0$$

$$\boxed{\frac{a-c}{b(N+1)} = q^*}$$

$$EN = \left(q_1 = \frac{a-c}{b(N+1)}, q_2 = \frac{a-c}{b(N+1)}, \dots, q_N = \frac{a-c}{b(N+1)}\right)$$

$$\sum q^* = Q^* = \frac{(a-c)N}{b(N+1)}$$

$$P = a - bQ = \boxed{a - \frac{(a-c)N}{N+1}}$$

$$\Pi^j = \left(a - \frac{(a-c)N}{N+1} - c\right)\left(\frac{a-c}{b(N+1)}\right)$$

$$\Pi^j = \left(\frac{a(a-c)}{b(N+1)} - \frac{(a-c)^2 N}{b(N+1)^2} - \frac{c(a-c)}{b(N+1)}\right)$$

Cournot Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

Cournot Competition

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$$q^* = \frac{a-c}{b(N+1)}$$

Cournot Competition

- The FOC for a given firm is:

$$a - b\sum_{i=1}^{N} q^i - bq_j - c = 0$$

- The symmetric Nash equilibrium is given by

$$q^* = \frac{a-c}{b(N+1)}$$

- Thus

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$
$$p = a - N\frac{a-c}{(N+1)} < a$$
$$\Pi^j = \frac{(a-c)^2}{b(N+1)^2}$$

Cournot Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$
$$p = a - N\frac{a-c}{(N+1)} < a$$
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- As $N \to \infty$ we get close to perfect competition

Cournot Competition

$$\sum_{j=1}^{N} q^j = \frac{N(a-c)}{b(N+1)}$$
$$p = a - N\frac{a-c}{(N+1)} < a$$
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- As $N \to \infty$ we get close to perfect competition
- $N = 1$ we get the monopoly case

$\boxed{N = 1}$ Monopolio

$$q^* = \frac{a-c}{2b}$$

$$P = a - \frac{(a-c)}{2} = \frac{a+c}{2}$$

$$\Pi = \left(\frac{a+c}{2} - c\right)\left(\frac{a-c}{2b}\right)$$

$$= \left(\frac{a-c}{2}\right)\left(\frac{a-c}{2b}\right) = \frac{(a-c)^2}{4b}$$

$\boxed{N \to \infty}$ "Competencia Perfecta"

Lecture 13: Game Theory // Nash equilibrium

Examples - Continued

Examples - Continued

Cournot - Revisited

Bertrand Competition

Bertrand Competition - Different costs

Bertrand Competition - 3 Firms

Hotelling and Voting Models

## Bertrand Competition

- Consider the alternative model in which firms set prices
- In the monopolist's problem, there was not distinction between a quantity-setting model and a price setting
- In oligopolistic models, this distinction is very important

## Bertrand Competition

- Consider two firms with the same marginal constant marginal cost of production and demand is completely inelastic
- Each firm simultaneously chooses a price $p_i \in [0, +\infty)$
- If $p_1, p_2$ are the chosen prices, then the utility functions of firm $i$ is given by:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 0 & \text{if } p_i > p_{-i}, \\ (p_i - c)\frac{Q(p_i)}{2} & \text{if } p_i = p_{-i}, \\ (p_i - c)Q(p_i) & \text{if } p_i < p_{-i}. \end{cases}$$

## Bertrand Competition

- Assume that the marginal revenue function is strictly decreasing ($MR'(p_i) < 0$):

$$\begin{aligned} R(p_i) &= p_i Q(p_i) && (1) \\ MR(p_i) &= Q(p_i) + p_i Q'(p_i) && (2) \\ &= Q(p_i)\left(1 + \varepsilon_{Q,p}(p_i)\right). && (3) \end{aligned}$$

## Bertrand Competition

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- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.

## Bertrand Competition

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- Let $p^m > c \geq 0$ be the monopoly price such that $MR(p^m) = c$.
- Then

$$MR(p_i) - c > 0 \text{ if } p_i < p^m, MR(p_i) - c < 0 \text{ if } p_i > p^m.$$

## Bertrand Competition

- The best response function is:

$$BR_i(p_{-i}) = \begin{cases} p^m & \text{if } p_{-i} > p^m, \\ p_{-i} - \varepsilon & \text{if } c < p_{-i} \leq p^m, \\ [c, +\infty) & \text{if } c = p_{-i} \\ (c, +\infty) & \text{if } c > p_{-i}. \end{cases}$$

- Where $\varepsilon$ is the smallest monetary unit

2 Firmas

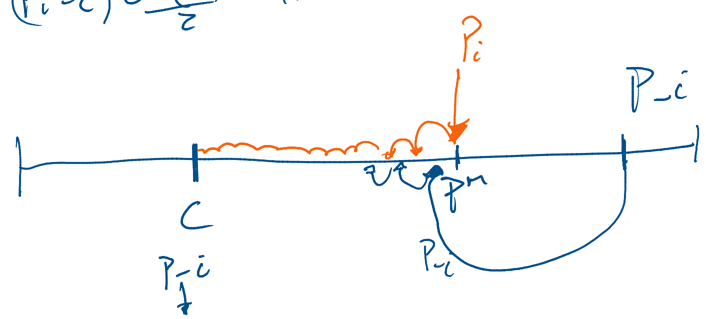
$cmg = c$

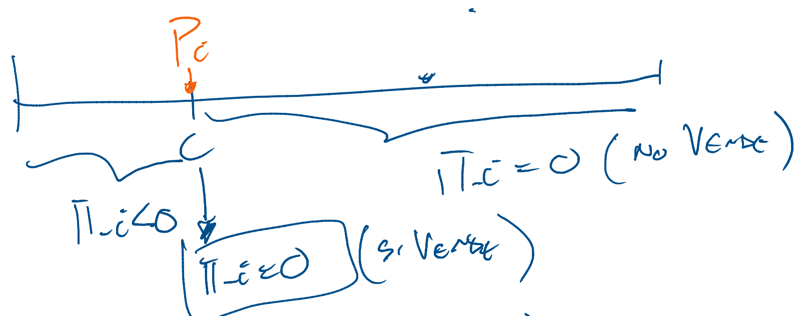
$P_i < P_{-i}$ → Todos le compran a "i"

$P_{-i} < P_i$ → Todos compran a "-i"

$P_{-i} = P_i$ → Se reparten el mcdo

$$\Pi_i = \begin{cases} 0 & P_i > P_{-i} \\ (P_i - c)\,Q_i(P_i) & P_i < P_{-i} \\ (P_i - c)\,\frac{Q_i(P_i)}{2} & P_i = P_{-i} \end{cases}$$




$$EN = (P_1 = c,\ P_2 = c)$$

$$\lim_{N \to \infty} q_j = \frac{a-c}{b(N+1)} = 0$$

$$\lim_{N \to \infty} Q^* = \frac{(a-c)N}{b(N+1)} = \frac{a-c}{b}$$

$$\lim_{N \to \infty} P = a - \frac{(a-c)N}{N+1} = a - \frac{(a-c)}{1} = c$$

$$\lim_{N \to \infty} \Pi_j = 0$$

## Bertrand Competition

**Case 1:** $p_1^* > p^m$

- $p_2^* = p^m$

## Bertrand Competition

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- $BR_2(p^m) = p^m - \varepsilon$

## Bertrand Competition

**Case 1:** $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$
- $BR_1(p^m - \varepsilon) = p^m - 2\varepsilon$

## Bertrand Competition

**Case 1:** $p_1^* > p^m$

- $p_2^* = p^m$
- $BR_2(p^m) = p^m - \varepsilon$
- $BR_1(p^m - \varepsilon) = p^m - 2\varepsilon$
- So this cannot be a Nash equilibrium

## Bertrand Competition

**Case 2:** $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$

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**Case 2:** $p_1^* \in (c, p^m]$

- $BR_2(p_1^*) = p_1^* - \varepsilon$
- $BR_1(p_1^* - \varepsilon) = p_1^* - 2\varepsilon$
- So this cannot be a Nash equilibrium

Bertrand Competition

**Case 3:** $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$

Bertrand Competition

**Case 3:** $p_1^* < c$

- $BR_2(p_1^*) \in [p_1^* + \varepsilon, \infty)$
- So this cannot be a Nash equilibrium

Bertrand Competition

**Case 4:** $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$

Bertrand Competition

**Case 4:** $p_1^* = c$

- $BR_2(p_1^*) = (c, +\infty)$
- The unique pure strategy Nash equilibrium is $p_1^* = p_2^* = c$

Bertrand Competition

Thus in contrast to the Cournot duopoly model, in the Bertrand competition model, two firms get us back to perfect competition ($p = c$)

Lecture 13: Game Theory // Nash equilibrium



Bertrand Competition - different costs

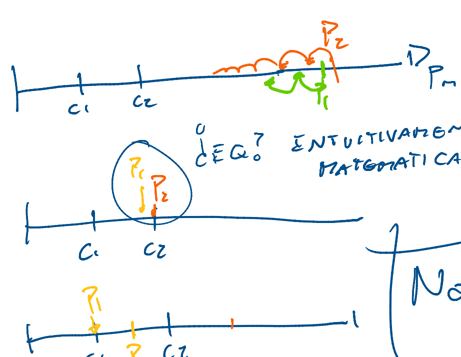
- Suppose that the marginal cost of firm 1 is equal to $c_1$ and the marginal cost of firm 2 is equal to $c_2$ where $c_1 < c_2$.
- The best response for each firm:

$$BR_i(p_{-i}) = \begin{cases} p_m^i & \text{if } p_{-i} > p_m^i, \\ p_{-i} - \varepsilon & \text{if } c_i < p_{-i} \leq p_m^i, \\ [c_i, +\infty) & \text{if } p_{-i} = c_i \\ (p_{-i}, +\infty) & \text{if } p_{-i} < c_i. \end{cases}$$

Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$ , then firm 2 would be making a loss

$c_1 < c_2$




¿EQ? INTUITIVAMENTE SI, MATEMATICAMENTE NO

NO TIENE E.N.

PRECIOS DISCRETOS

## Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$ , then firm 2 would be making a loss

## Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$ , then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$ , then firm 1 would cut prices to keep the whole market

## Bertrand Competition - different costs

- If $p_2^* = p_1^* = c_1$ , then firm 2 would be making a loss
- If $p_2^* = p_1^* = c_2$ , then firm 1 would cut prices to keep the whole market
- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut $p_2^*$ and get a positive profit

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- Firm 1 would really like to price at some price $p_1^*$ just below the marginal cost of firm 2, but wherever $p_2$ is set, Firm 1 would try to increase prices

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- If $p_2^* = p_1^* = c_1$ , then firm 2 would be making a loss
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- Any pure strategy NE must have $p_2^* \leq c_1$. Otherwise, if $p_2^* > c_1$ then firm 1 could undercut $p_2^*$ and get a positive profit
- Firm 1 would really like to price at some price $p_1^*$ just below the marginal cost of firm 2, but wherever $p_2$ is set, Firm 1 would try to increase prices
- No NE because of continuous prices

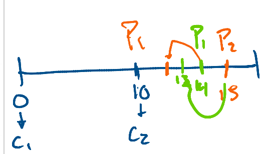
## Bertrand Competition - discreet prices

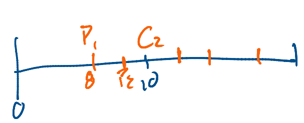
- Suppose $c_1 = 0 < c_2 = 10$

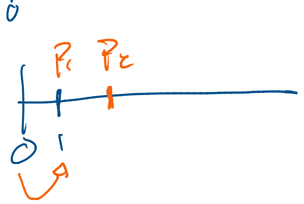
## Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.

Precios Discretos




EN = {(10, 11), (9, 10), (8, 9), (7, 8), (6, 7), (5, 6), (4, 5), (3, 4), (2, 3), (1, 2)}

## Bertrand Competition - discreet prices

- Suppose $c_1 = 0 < c_2 = 10$
- Firms can only set integer prices.
- Suppose that $(p_1^*, p_2^*)$ is a pure strategy Nash equilibrium...

## Bertrand Competition - discreet prices

**Case 1:** $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$

## Bertrand Competition - discreet prices

**Case 1:** $p_1^* = 0$

- Best response of firm 2 is to choose some $p_2^* > p_1^*$
- $p_1^*$ cannot be a best response to $p_2^*$ since by setting $p_1 = p_2^*$ firm 1 would get strictly positive profits

## Bertrand Competition - discreet prices

**Case 2:** $p_1^* \in \{1, 2, \dots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$

## Bertrand Competition - discreet prices

**Case 2:** $p_1^* \in \{1, 2, \dots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price

## Bertrand Competition - discreet prices

**Case 2:** $p_1^* \in \{1, 2, \dots, 9\}$

- Best response of firm 2 is to set any price $p_2^* > p_1^*$
- If $p_2^* > p_1^* + 1$, then this cannot be a Nash equilibrium since then firm 1 would have an incentive to raise the price
- The only equilibrium is $(p_1^*, p_1^* + 1)$

## Bertrand Competition - discreet prices

**Case 3:** $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$

## Bertrand Competition - discreet prices

**Case 3:** $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

## Bertrand Competition - discreet prices

**Case 3:** $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher

## Bertrand Competition - discreet prices

**Case 3:** $p_1^* = 10$

- Best responses of firm 2 is to set any price $p_2^* \geq p_1^*$
- It cannot be that $p_2^* = p_1^*$ since then firm 1 would rather deviate to a price of 9 and control the whole market:

$$\frac{1}{2}(10) = 5 < 9.$$

- We must have $p_2^* = p_1^* + 1$ since otherwise, firm 1 would have an incentive to raise the price higher
- $(p_1^*, p_2^*) = (10, 11)$ is a Nash equilibrium

## Bertrand Competition - discreet prices

**Case 4:** $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$

## Bertrand Competition - discreet prices

**Case 4:** $p_1^* = 11$

- Best response of firm 2 is to set $p_2^* = 11$
- Firm 1 would not be best responding since by setting a price of $p_1 = 10$, it would get strictly positive profits

## Bertrand Competition - discreet prices

**Case 5:** $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$

## Bertrand Competition - discreet prices

**Case 5:** $p_1^* \geq 12$

- Firm 2's best response is to set either $p_2^* = p_1^* - 1$ or $p_2^* = p_1^*$
- Firm 1 is not best responding since by lowering the price it can get the whole market.

## Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms

## Bertrand Competition - 3 firms

PREZZO CONTINUO

- Symmetric marginal costs model but with 3 firms
- Best response of firm $i$ is given by:

$$BR_1(p_2, p_3) = \begin{cases} p^m & \text{if } \min\{p_2, p_3\} > p^m, \\ \min\{p_2, p_3\} - \varepsilon & \text{if } c < \min\{p_2, p_3\} \leq p^m, \\ [c, +\infty) & \text{if } c = \min\{p_2, p_3\}, \\ (\min\{p_2, p_3\}, +\infty) & \text{if } c > \min\{p_2, p_3\}. \end{cases}$$

## Bertrand Competition - 3 firms

- Symmetric marginal costs model but with 3 firms
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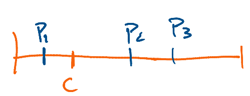
- $(c, c, c)$ is indeed a pure strategy Nash equilibrium as in the two firm case

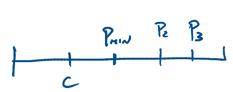
## Bertrand Competition - 3 firms

- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$



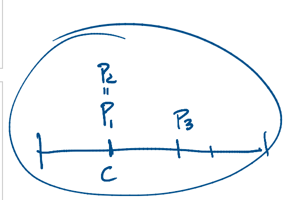

## Bertrand Competition - 3 firms

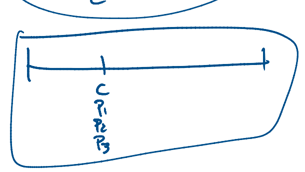
- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$




## Bertrand Competition - 3 firms

- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} > c$
- We must have $\min\{p_1, p_2, p_3\} = c$




## Bertrand Competition - 3 firms

- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min\{p_1, p_2, p_3\} < c$
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## Bertrand Competition - 3 firms

- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min(p_1, p_2, p_3) < c$
- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min(p_1, p_2, p_3) > c$
- We must have $\min(p_1, p_2, p_3) = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to $c$?


## Bertrand Competition - 3 firms

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- We must have $\min(p_1, p_2, p_3) = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to $c$? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost

## Bertrand Competition - 3 firms

- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min(p_1, p_2, p_3) < c$
- If $(p_1, p_2, p_3)$ was a pure strategy Nash equilibrium, it can never be the case that $\min(p_1, p_2, p_3) > c$
- We must have $\min(p_1, p_2, p_3) = c$
- Can there be a pure strategy Nash equilibrium in which just one firm sets price equal to $c$? No since that firm would want to raise his price a bit and get strictly better profits
- There must be at least two firms that set price equal to marginal cost
- Set of all pure strategy Nash equilibria are given by:

$$\{(c, c, c+\varepsilon) : \varepsilon \geq 0\} \cup \{(c, c+\varepsilon, c) : \varepsilon \geq 0\} \cup \{(c+\varepsilon, c, c) : \varepsilon \geq 0\}.$$

## Lecture 13: Game Theory // Nash equilibrium



## Hotelling

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- If the firms $i = 1, 2$ respectively produce products of characteristic $x_1$ and $x_2$, then a consumer at $\theta$ would consume whichever product is closest to $\theta$
- The game consists of the two players $i = 1, 2$, each of whom chooses a point $x_1, x_2 \in [0, 1]$ simultaneously.

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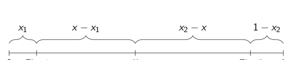


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Then the profits that accrue to firm 1 is given by the mass of consumers that are closest to firm 1:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

Similarly,

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1+x_2}{2} & \text{if } x_1 < x_2, \\ \frac{1}{2} & \text{if } x_1 = x_2, \\ \frac{x_1+x_2}{2} & \text{if } x_1 > x_2. \end{cases}$$

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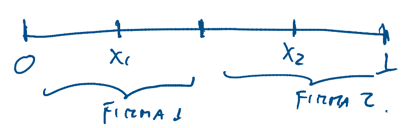


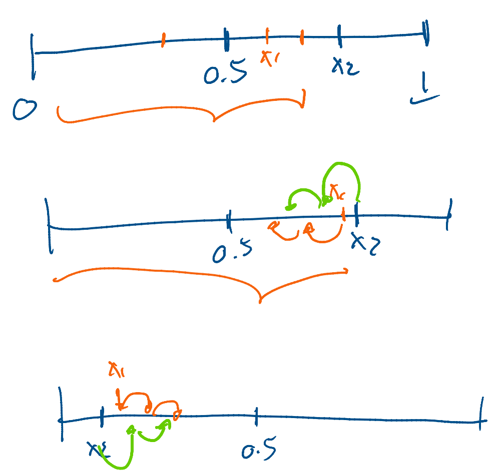


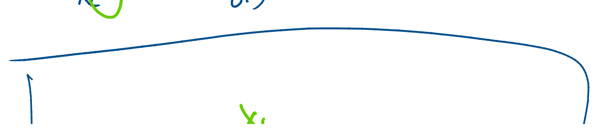

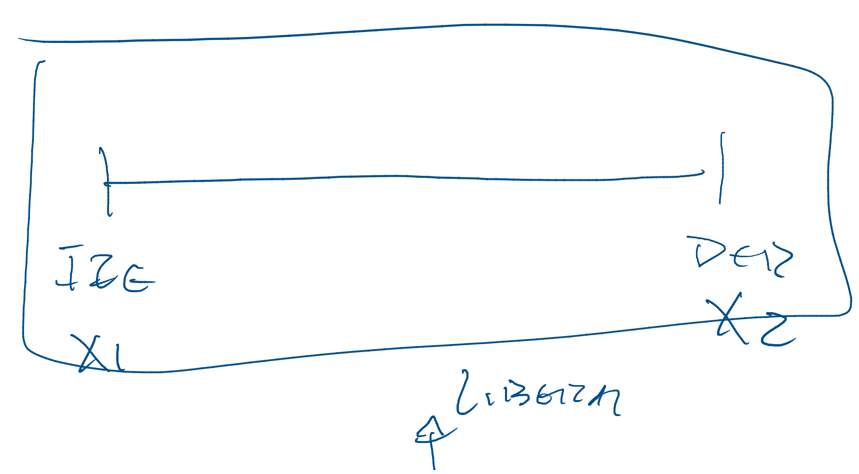

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Compute the best response functions

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- **Case 2:** Suppose next that $x_2 < 1/2$. Again there will be no best response for firm 1 (try to set as close to the right the other firm as possible)
- **Case 3:** Suppose next that $x_2 = 1/2$. Here there will be a best response for firm 1 at $1/2$

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$$BR_1(x_2) = \begin{cases} \emptyset & \text{if } x_2 > 1/2 \\ 1/2 & \text{if } x_2 = 1/2 \\ \emptyset & \text{if } x_2 < 1/2. \end{cases}$$

Symmetrically, we have:

$$BR_2(x_1) = \begin{cases} \emptyset & \text{if } x_1 > 1/2 \\ 1/2 & \text{if } x_1 = 1/2 \\ \emptyset & \text{if } x_1 < 1/2. \end{cases}$$

The unique Nash equilibrium is for each firm to choose $(x_1, x_2) = (1/2, 1/2)$. Each firm essentially locates in the same place

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- Hotelling can also be done in a discreet setting
- Hotelling can be applied to a variety of situations (e.g., voting)
- But this predicts the opposite of polarization
- With three candidates, predictions are quite different
- All candidates picking $\frac{1}{2}$ is no longer a Nash equilibrium
- What are the set of pure strategy equilibria here? (this is a difficult problem).