Lecture2

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Lecture2

Lecture 2: General Equilibrium

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Lecture 2: General Equilibrium

Cobb-Douglas
Using calculus
Perfect substitutes
Perfect complements

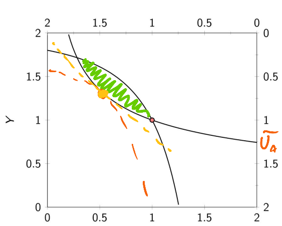
Cobb-Douglas

$$u_A(x, y) = x^{\alpha} y^{1-\alpha}$$
$$u_B(x, y) = x^{\beta} y^{1-\beta}$$

For graph suppose

$$\alpha = 0.7$$
$$\beta = 0.3$$
$$\omega^A = (1, 1)$$
$$\omega^B = (1, 1)$$

Cobb-Douglas



Cobb-Douglas

- Indifference curves must be tangent (formalize this later)

$$\rightarrow \frac{\partial U(x,y)}{\partial x} = \frac{\text{utiles}}{x}$$

- Thus, the MRS must be equalized across the two consumers

$$\frac{\frac{\text{utiles}}{x}}{\frac{\text{utiles}}{y}} = MRS^A_{x,y} = \frac{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial x}}{\frac{\partial x^{\alpha} y^{1-\alpha}}{\partial y}} = \frac{\alpha}{1-\alpha} \frac{x^{\alpha-1} y^{1-\alpha}}{x^{\alpha} y^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{y^A}{x^A} \rightarrow \text{TMS}$$

$$MRS^B_{x,y} = \frac{\frac{\partial x^{\beta} y^{1-\beta}}{\partial x}}{\frac{\partial x^{\beta} y^{1-\beta}}{\partial y}} = \frac{\beta}{1-\beta} \frac{x^{\beta-1} y^{1-\beta}}{x^{\beta} y^{-\beta}} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

$$\frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \frac{y^B}{x^B}$$

Aparte

$$x^A + x^B = 2 \qquad y^A + y^B = 2$$

$$\rightarrow x^B = 2 - x^A$$
$$\rightarrow y^B = 2 - y^B$$

$$\rightarrow \frac{\alpha}{1-\alpha} \frac{y^A}{x^A} = \frac{\beta}{1-\beta} \left(\frac{2 - y^A}{2 - x_A} \right)$$

$$(2 - x_A) y^A = \frac{\beta(1-\alpha)}{(1-\beta)\alpha} \cdot (2 - y^A) x^A$$

$$2y^A - x_A y_A = 2 \frac{\beta}{\alpha} \frac{1-\alpha}{1-\beta} x^A - \frac{\beta}{\alpha} \frac{(1-\alpha)}{(1-\beta)} y^A x^A$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$
$$y^A + y^B = \omega_y$$

$(1-\beta)\alpha$

$$2y^A - x_A y_A = 2\frac{\beta}{\alpha}\frac{1-\alpha}{1-\beta}x^A - \frac{\beta}{\alpha}\frac{(1-\alpha)}{(1-\beta)}y^A x^A$$

$$2y^A - x_A y_A + \frac{\beta}{\alpha}\frac{(1-\alpha)}{1-\beta}y^A x^A = 2\frac{\beta}{\alpha}\frac{1-\alpha}{1-\beta}x^A$$

$$y^A\left(2 - x_A + \frac{\beta}{\alpha}\frac{1-\alpha}{1-\beta}x_A\right) = \frac{2\beta}{\alpha}\frac{1-\alpha}{1-\beta}x^A$$

$$y^A = \frac{\frac{2\beta}{\alpha}\frac{1-\alpha}{1-\beta}x^A}{2 - x_A + \frac{\beta}{\alpha}\frac{1-\alpha}{1-\beta}x_A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

Cobb-Douglas

But we haven't used the fact that

$$x^A + x^B = \omega_x$$

$$y^A + y^B = \omega_y$$

$$\frac{\alpha}{1-\alpha}\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\frac{\omega_y - y^A}{\omega_x - x^A}$$

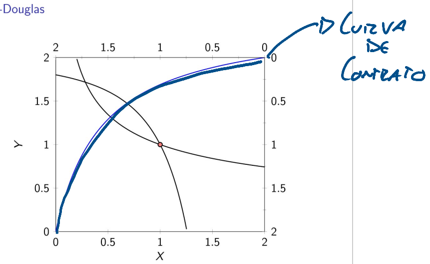
$$y^A = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta}\left(\frac{\omega_y - y^A}{\omega_x - x^A}\right)$$

$$y^A\left(1 + \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{x^A}{\omega_x - x^A}\right) = x^A \cdot \frac{1-\alpha}{\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\omega_y}{\omega_x - x^A}$$

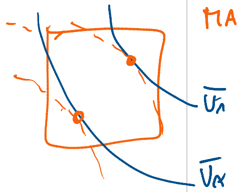
Then:

$$y^A = \frac{(1-\alpha)\beta\omega_y x^A}{\alpha w_x - \alpha x^A - \alpha\beta w_x + \beta x^A}$$

Cobb-Douglas



- Cobb-Douglas
- Using calculus
- Perfect substitutes
- Perfect complements



MAX U_B s.a $U_A \geq \bar{U}_A$

$X_A + X_B \leq \omega_x^A + \omega_x^B$

$y_A + y_B \leq \omega_y^A + \omega_y^B$ } Factible

Essentially in this exercise we are doing the following:

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A,y^A) \text{ such that}$$

$$u_B(x^B,y^B) \geq \underline{u}_B$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

Theorem

Consider an Edgeworth Box economy and suppose that all consumers have strictly monotone utility functions. Then a feasible allocation $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient if and only if it solves

$$\max_{(x^A,y^A),(x^B,y^B)} u_A(x^A,y^A) \text{ such that}$$

$$u_B(x^B,y^B) \geq \underline{u}_B$$
$$x^B + x^A \leq \omega_x,$$
$$y^B + y^A \leq \omega_y.$$

$x^B = \omega_x - x^A$

$y^B = \omega_y - y^A$

$$\boxed{\begin{aligned}\mathcal{L} = {} & U_A(x_A, x_B) \\ & + \lambda_1 \left(U_B(x_B, y_B) - \bar{U}_B\right) \\ & + \lambda_2 \left(\omega_x - x_A - x_B\right) \\ & + \lambda_3 \left(\omega_y - y_A - y_B\right)\end{aligned}}$$

$$\mathcal{L} = U_A(x_A, x_B) + \lambda_1\left(U_B(\omega_x - x_A, \omega_y - y_A) - \bar{U}_B\right)$$

- Very tempting to use lagrangeans, no?
- We need to assume all consumers have quasi-concave, strictly monotone, differentiable utility functions

Then we can solve:

$$\mathcal{L} = u_A(x^A,y^A) + \lambda(u_B(\omega_x - x^A, \omega_y - y^A) - \underline{u}_B)$$

$$\frac{\partial \mathcal{L}}{\partial x_A} = \frac{\partial U_A(x_A,y_A)}{\partial x_A} + \lambda \frac{\partial U_B(x_B,y_B)}{\partial x_B} \cdot (-1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_A} = \frac{\partial U_A(x_A,y_A)}{\partial y_A} + \lambda \frac{\partial U_B(x_B,y_B)}{\partial y_B} (-1) = 0$$

Lets take the first order conditions of the above problem. Beginning with X^A:

$$\frac{\partial \mathcal{L}}{\partial x^A} : \frac{\partial u_A}{\partial x}(x^A,y^A) - \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial x}(x^{A*},y^{A*}) = \lambda \frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

For y^A:

$$\frac{\partial \mathcal{L}}{\partial y^A} : \frac{\partial u_A}{\partial y}(x^A,y^A) - \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^A, \omega_y - y^A) = 0$$

which implies:

$$\frac{\partial u_A}{\partial y}(x^{A*},y^{A*}) = \lambda \frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})$$

$$\frac{\frac{\partial U_A}{\partial x_A}(x_A, y_A)}{\frac{\partial U_A}{\partial y_A}(x_A, y_A)} = \frac{\frac{\partial U_B}{\partial x_B}}{\frac{\partial U_B}{\partial y_B}}$$

$$TMS_A = TMS_B$$

If $(x^{A*}, y^{A*}, x^{B*}, y^{B*})$ is Pareto efficient then

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*},y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*},y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*},y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*},y^{B*})}.$$

- In short $MRS^A_{x,y} = MRS^B_{x,y}$
- This condition is *necessary* and *sufficient*

Theorem

Suppose that both consumers have utility functions that are quasi-concave and strictly increasing. Suppose that $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is an **interior** *feasible allocation. Then $(x^{A*}, y^{A*}, \omega_x - x^{A*}, \omega_y - y^{A*})$ is Pareto efficient if and only if*

$$\frac{\frac{\partial u_A}{\partial x}(x^{A*}, y^{A*})}{\frac{\partial u_A}{\partial y}(x^{A*}, y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(\omega_x - x^{A*}, \omega_y - y^{A*})}{\frac{\partial u_B}{\partial y}(\omega_x - x^{A*}, \omega_y - y^{A*})} = \frac{\frac{\partial u_B}{\partial x}(x^{B*}, y^{B*})}{\frac{\partial u_B}{\partial y}(x^{B*}, y^{B*})}.$$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

A: $-1y$, $+1x$

B: $+1y$, $-1x$

Intuition

Suppose that we are at an allocation where $MRS^A_{x,y} = 2 > MRS^B_{x,y} = 1$. Can we make both consumers better off?

- A gives up 1 unit of y to person B in exchange for unit of x
- B is indifferent since his $MRS^B_{x,y} = 1$.
- A receives a unit of x and only needs to give one unit of y (he was willing to give two)
- We have reallocated goods to make A strictly better off without hurting B

General case

$$\max_{((x_1^1,\dots,x_L^1),\dots,(x_1^I,\dots,x_L^I))} u_1(x_1^1,\dots,x_L^I) \text{ such that } u_2(x_1^2,\dots,x_L^2) \geq \underline{u}_2,$$
$$\vdots$$
$$u_I(x_1^I,\dots,x_L^I) \geq \underline{u}_I,$$
$$x_1^1 + \cdots + x_1^I \leq \omega_1,$$
$$\vdots$$
$$x_L^1 + \cdots + x_L^I \leq \omega_L.$$

General case

Theorem

Suppose that all utility functions are strictly increasing and quasi-concave. Suppose also that $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is a feasible interior allocation. Then $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ is Pareto efficient if and only if $((\hat{x}_1^1,\dots,\hat{x}_L^1),\dots,(\hat{x}_1^I,\dots,\hat{x}_L^I))$ exhausts all resources **and** *for all pairs of goods ℓ, ℓ',*

$$MRS^1_{\ell,\ell'}(\hat{x}_1^1,\dots,\hat{x}_L^1) = \cdots = MRS^I_{\ell,\ell'}(\hat{x}_1^I,\dots,\hat{x}_L^I).$$

X, Y, Z

AGENTES: A, B, C

$TMS^A_{x,y} = TMS^B_{x,y} = TMS^C_{x,y}$

$TMS^A_{x,z} = TMS^B_{x,z} = TMS^C_{x,z}$

$TMS^A_{y,z} = TMS^B_{y,z} = TMS^C_{y,z}$

- Utility functions must be strictly increasing, quasi-concave, and differentiable!

Lecture 2: General Equilibrium

- Cobb-Douglas
- Using calculus
- Perfect substitutes
- Perfect complements

Perfect substitutes

Suppose that

$$u_A(x^A, y^A) = 2x^A + y^A$$
$$u_B(x^B, y^B) = x^B + y^B$$
$$\omega^A = (1,1)$$
$$\omega^B = (1,1)$$

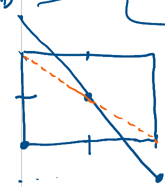
$$\cancel{\mathcal{L} = 2x^A + y^A + \lambda\left(\bar{U}_B - \underbrace{(2-x^A)}_{x^B} - \underbrace{(2-y^A)}_{y^B}\right)}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial x^A} = 2 + \lambda = 0}$$

$\bar{U}_A = 2x^A + y^A \rightarrow y^A = \bar{U}_A - 2x_A$

$U_A = 2 \cdot 1 + 1 = 3$

$y^A = 3 - 2x_A$



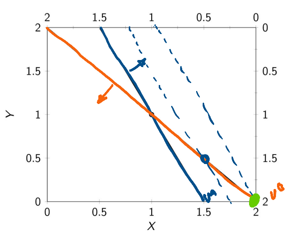
$\bar{U}_B = x^B + y^B \rightarrow y^B = \bar{U}_B - x_B$

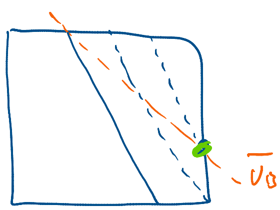
$2 - y^A = \bar{U}_B - (2 - x^A)$

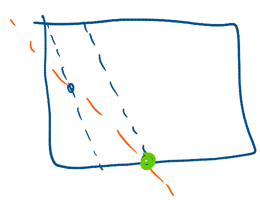
$2 + (2 - x^A) - \bar{U}_B = y^A$

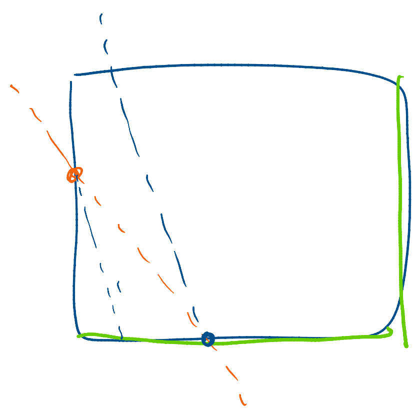
$4 - \underline{x^A} - \bar{U}_B = y^A$

Perfect substitutes

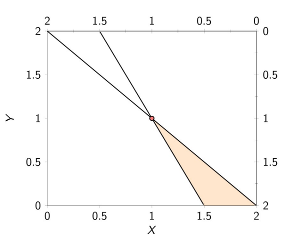




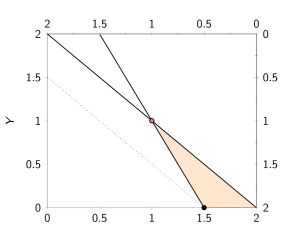




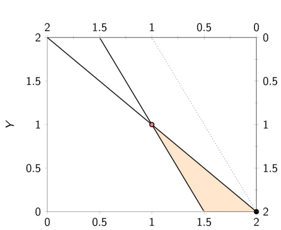
Perfect substitutes



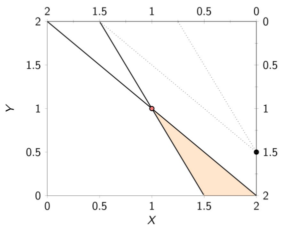
Perfect substitutes



Perfect substitutes



Perfect substitutes

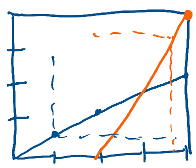


Lecture 2: General Equilibrium

Perfect complements

Suppose that

$$u_A(x^A, y^A) = \min(x^A, 2y^A)$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (3, 1)$$
$$\omega^B = (1, 3)$$



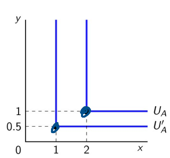
↳ $x^A = 2y^A$ ↳ $y^A = \frac{x^A}{2}$

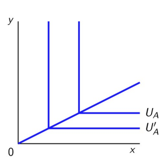
$2x^B = y^B$

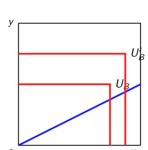
↳ $2(4-x^A) = 4-y^A$

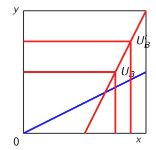
$= 8-2x^A = 4-y^A$

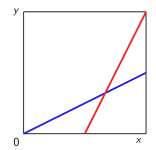
$\boxed{y^A = 2x^A - 4}$

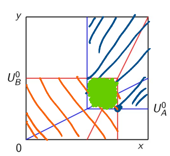


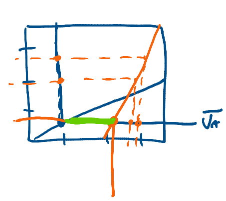


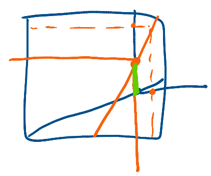




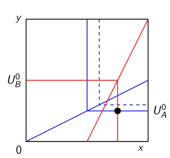




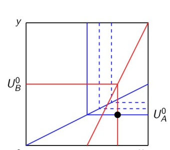




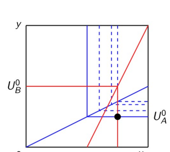
Make A as well as we can without making B worse off



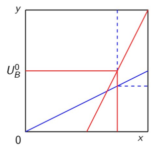
Make A as well as we can without making B worse off



Make A as well as we can without making B worse off



Make A as well as we can without making B worse off



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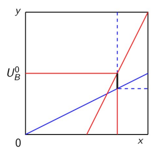
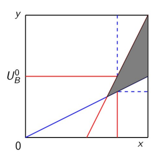
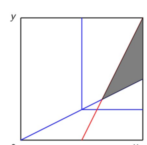
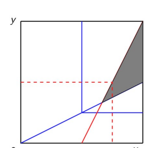
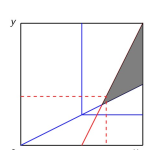


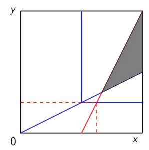
Make A as well as we can without making B worse off

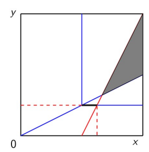


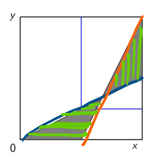


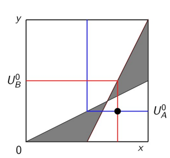


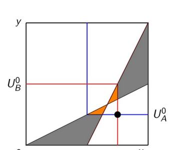












- What about: $u_A(x, y) = x^2 + y^2, u_B(x, y) = x + y$?

- Try it at home!

Recap

- We expect all exchanges to happen on the contract curve (hence its name)
- We expect all **voluntary** exchanges to be in the orange box
- Can we say more? Not without prices