Lecture 4: General Equilibrium

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Lecture 4: General Equilibrium

Is there always an equilibrium?

Is the equilibrium unique?

First welfare theorem

Second welfare theorem

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- The answer is going to be yes in general

- We will show that the equilibrium is a “fix point” of a certain function

- Intuitively, if we have a function that adjusts prices (higher price is demand $>$ supply), then the equilibrium is where this function stops updating

Is there always an equilibrium?

- An intro to fix point theorems
- The walrasian auctioneer

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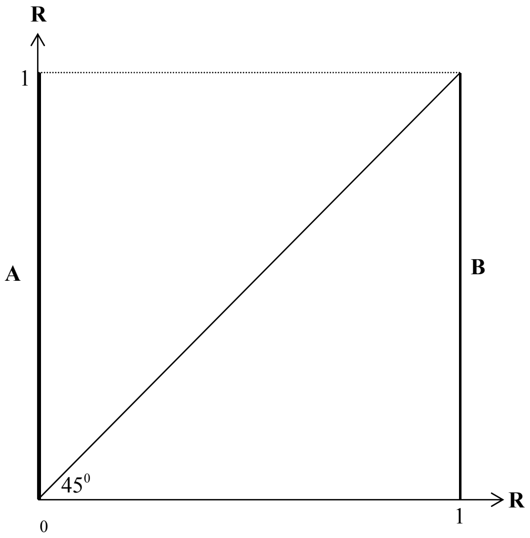
Try to draw a line from A to B without crossing the diagonal



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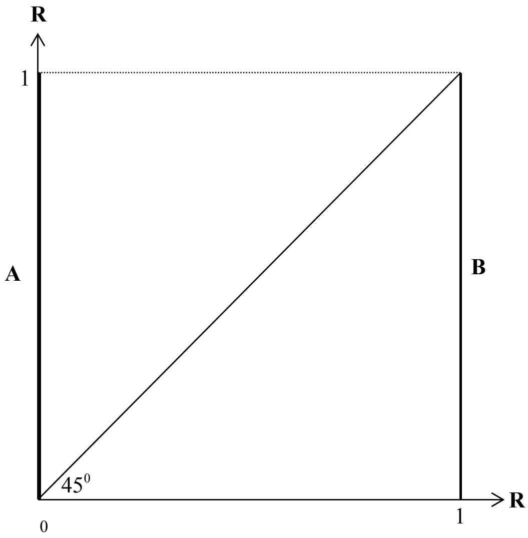
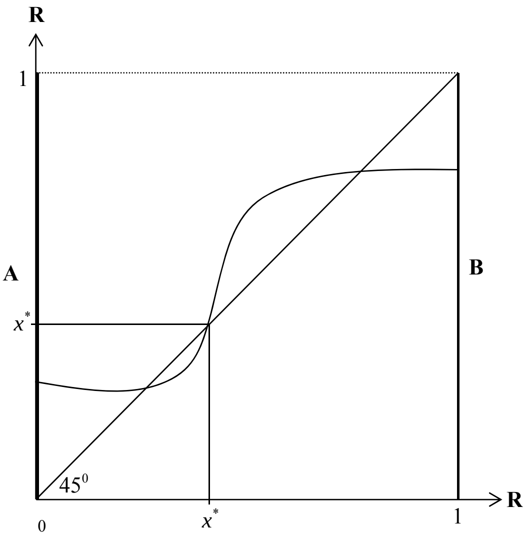


Its impossible!

For example...



There is even a theorem for this:

Theorem

For any function $f : [0,1] \to [0,1]$ *that is continuous, there exists an* $x^* \in [0,1]$ *such that* $f(x^*) = x^*$

And a more general version!

Theorem

For any function $f : \triangle^{L-1} \to \triangle^{L-1}$ that is continuous, there exists a point $p^* = (p_1^*, p_2^*, ..., p_L^*)$ such that

$$f(p^*) = p^*.$$

where

$$\triangle^{L-1} = \{(p_1, p_2, ..., p_L) \in \mathbb{R}_+^L \mid \sum_{l=1}^{L} p_l = 1\}$$

What was the goal again?

- Prove the existence of a general equilibrium in a market
- We will show that the equilibrium is a "fix point" of a certain function
- Intuitively, if we have a function that adjusts prices (higher price if demand $>$ supply), then the equilibrium is where this function stops updating

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Excess demand

Let us define the excess demand by:

$$Z(p) = (Z_1(p), Z_2(p), ..., Z_L(p)) = \sum_{i=1}^{I} {x^*}^i(p) - \sum_{i=1}^{I} w^i$$

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since $x^{*i}(p)$ is the demand (i.e., consumers are already maximizing) then we have the following result:

Remark

$p \in \mathbb{R}^L_{++}$ is a competitive equilibrium if and only if $Z(p) = 0$

Excess demand

Z(p) has the following properties

1. Is continuous in p
2. Is homogeneous of degree zero
3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law)

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1. Is continuous in p
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3. $p \cdot Z(p) = 0$ (this is equivalent to Walra's law) — Think about this!

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$$p' = p + Z(p)$$

But what if $p' < 0$? Ok then

$$T(p) = \frac{1}{\sum_{i=1}^{L} p_l + max\left(0, Z_l(p)\right)} \left(p_1 + \max\left(0, Z_1(p)\right),\right.$$
$$p_2 + \max\left(0, Z_2(p)\right), \ldots,$$
$$\left. p_L + \max\left(0, Z_L(p)\right)\right)$$

Excess demand

- T is continuous
- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $T(p^*) = p^*$
- Then $Z(p^*) = 0$

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- Thus we can apply the fix point theorem
- Therefore there exists a p^* such that $T(p^*) = p^*$
- Then $Z(p^*) = 0$ (why?)

So when does it break down?

- We needed demand to be continuous!

Weird case - no equilibrium

$$\begin{aligned} u_A(x^A, y^A) &= \min(x^A, y^A) \\ u_B(x^B, y^B) &= \max(x^B, y^B) \\ \omega^A &= (1, 1) \\ \omega^B &= (1, 1) \end{aligned}$$

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- if $p_y > 1$ then B wants to demand as much of x as possible $X^b = p_y + 1$
- if $p_y = 1$ then B either demands two units of X or two units of Y, but A demands at least one unit of each good

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We have seen it is not

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Theorem

Consider any pure exchange economy. Suppose that all consumers have weakly monotone utility functions. Then if (x^*, p) *is a competitive equilibrium, then* x^* *is a Pareto efficient allocation.*

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Assume that $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ is a competitive equilibrium but that $\left(x^1, x^2, ..., x^I\right)$ is not Pareto efficient

Proof

By contradiction:
Assume that $(p, (x^1, x^2, ..., x^I))$ is a competitive equilibrium but that $(x^1, x^2, ..., x^I)$ is not Pareto efficient
Then there is an allocation $(\widehat{x}^1, \widehat{x}^2, ..., \widehat{x}^I)$ such that

- is feasible
- pareto dominates $(x^1, x^2, ..., x^I)$

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In other words:

1. $\sum_{i=1}^{I} \widehat{x}^i = \sum_{i=1}^{I} w^i$
2. For all i, $u^i\left(\widehat{x}^i\right) \geqslant u^i\left(x^i\right)$
3. For some i^*, $u^{i^*}\left(\widehat{x}^{i^*}\right) > u^{i^*}\left(x^{i^*}\right)$

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- Condition 3 in the previous slide implies $p \cdot \widehat{x}^{i^*} > p \cdot w^{i^*}$

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  - Otherwise, why didn't i^* pick $\widehat{x}^{i^*}$ to begin with
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Which contradicts what Condition 1 in the previous slide implies.

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- How about the opposite?
  - Maybe we "like" one Pareto allocation over another (for bio-ethic considerations)
  - Can any Pareto efficient allocation can be sustained as the outcome of some competitive equilibrium?
  - Not in general... but what if we allow for a redistribution of resources?

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Theorem

Given an economy $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, w^i\right)_{i \in \mathcal{I}} \right\rangle$ *where all consumers have weakly monotone, quasi-concave utility functions. If* $\left(x^1, x^2, ..., x^I\right)$ *is a Pareto optimal allocation then there exists a redistribution of resources* $\left(\widehat{w}^1, \widehat{w}^2, ..., \widehat{w}^I\right)$ *and some prices* $p = (p_1, p_2, ..., p_L)$ *such that:*

1. $\sum_{i=1}^{I} \widehat{w}^i = \sum_{i=1}^{I} w^i$
2. $\left(p, \left(x^1, x^2, ..., x^I\right)\right)$ *is a competitive equilibrium of the economy* $\mathcal{E} = \left\langle \mathcal{I}, \left(u^i, \widehat{w}^i\right)_{i \in \mathcal{I}} \right\rangle$

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  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution?

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- You **just** need to redistribute the endowments
  - Ok... but *what* re-distribution should I do to achieve a certain outcome? No idea
  - Ok... but *how* can we do this redistribution? Not taxes, since they produce dead-weight loss

- In contrast to the first welfare theorem, we require an additional assumption that all utility functions are quasi-concave.

- What if they are not? consider the following:

$$\begin{aligned} u_A(x, y) &= \max\{x, y\} \\ u_B(x, y) &= \min\{x, y\} \\ \omega^A &= (1, 1) \\ \omega^B &= (1, 1) \end{aligned}$$

In this example, all points in the Edgeworth Box are Pareto efficient. However we cannot obtain any of these points as a competitive equilibrium after transfers.