Mauricio Romero

Competitive equilibrium

Examples: Cobb-Douglas

Examples: Perfect Complements

Examples: Perfect Substitutes

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**Examples: Perfect Complements** 

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### Hidden assumptions

- There is a market for each good
- Every agent can access the market without any cost
- ► There is a unique price for each good and all consumers know this price
- ► Each consumer can sell her initial endowment in the market and use the income to buy goods and services
- Consumers seek to maximize their utility given their budget restriction, independently of what everyone else is doing.
  - ► There is no centralized mechanism
  - People may not know others preferences or endowments
- There is perfect competition (i.e., everyone is a price taker)
- ► The only source of information agents are prices



## Competitive equilibrium - Definition

#### Definition

A pair of an allocation and a price vector,  $(x^*, p = (p_1, \dots, p_L))$  is called a competitive equilibrium if the following conditions hold:

1. For all consumers  $i=1,2,\ldots,I$ ,  $x^{i^*}=(x_1^{i^*},\ldots,x_L^{i^*})$  solves the following maximization problem:

$$\max_{x^i} u_i(x^i)$$
 such that  $p \cdot x^i \leq p \cdot \omega^i = \sum_{\ell=1}^L p_\ell \omega^i_\ell.$ 

2. Markets clear: For each commodity  $\ell = 1, 2, ..., L$ , the following equation holds:

$$\sum_{i=1}^{I} x_{\ell}^{i*} = \sum_{i=1}^{I} \omega_{\ell}^{i}.$$

## Competitive equilibrium - Properties

#### Remark

Suppose that at least one consumer has **strictly** monotone preferences. Then if  $(x^*, p)$  is a competitive equilibrium,  $p_1, p_2, \ldots, p_L > 0$ .

#### Remark

Suppose that at least one consumer has **weakly** monotone preferences. Then if  $(x^*, p)$  is a competitive equilibrium, there for at least one  $\ell$ ,  $p_{\ell} > 0$ .

#### Remark

If  $(x^*, p)$  is a competitive equilibrium, then  $(x^*, cp)$  for  $c \in \mathbb{R}_+ +$  is also a competitive equilibrium.

## Competitive equilibrium - Walras' Law

## Theorem (Walras' Law)

Suppose that consumer i has weakly monotone preferences and that  $\hat{x}^i \in x^{i*}(p)$ . Then

$$p \cdot \hat{x}^i = \sum_{\ell=1}^L p_\ell \hat{x}^i_\ell = \sum_{\ell=1}^L p_\ell \omega^i_\ell = p \cdot \omega^i.$$

# Theorem (Walras' Law - II)

Suppose that utility functions are **weakly** monotonic. Suppose that  $p = (p_1, \ldots, p_L)$  is such that  $p_L > 0$ . Take any  $(x^*, p)$  in which Condition 1 holds for each consumer  $i = 1, 2, \ldots, I$  and markets clear for all commodities  $\ell = 1, 2, \ldots, L-1$ . Then the market clearing condition will hold for commodity L as well.

For each consumer *i*, we must

$$\sum_{\ell=1}^L p_\ell {x_\ell^i}^* = \sum_{\ell=1}^L p_\ell \omega_\ell^i.$$

For each consumer i, we must

$$\sum_{\ell=1}^{L} p_{\ell} x_{\ell}^{i*} = \sum_{\ell=1}^{L} p_{\ell} \omega_{\ell}^{i}.$$

▶ If we sum the above across all I consumers, then we get:

$$\sum_{i=1}^{I} \sum_{\ell=1}^{L} p_{\ell} x_{\ell}^{i*} = \sum_{i=1}^{I} \sum_{\ell=1}^{L} p_{\ell} \omega_{\ell}^{i}.$$

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Re-arranging:

$$\sum_{\ell=1}^{L} \sum_{i=1}^{I} p_{\ell} x_{\ell}^{i*} = \sum_{\ell=1}^{L} \sum_{i=1}^{I} p_{\ell} \omega_{\ell}^{i}.$$

 $\triangleright$  For each consumer i, we must

$$\sum_{\ell=1}^{L} p_{\ell} x_{\ell}^{i*} = \sum_{\ell=1}^{L} p_{\ell} \omega_{\ell}^{i}.$$

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Re-arranging:

$$\sum_{\ell=1}^{L} \sum_{i=1}^{I} p_{\ell} x_{\ell}^{i*} = \sum_{\ell=1}^{L} \sum_{i=1}^{I} p_{\ell} \omega_{\ell}^{i}.$$

► Re-arranging:

$$\sum_{i=1}^{L} p_{\ell} \sum_{i=1}^{I} \left( x_{\ell}^{i*} - \omega_{\ell}^{i} \right) = 0.$$

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$$p_L \sum_{i=1}^{I} \left( x_L^{i*} - \omega_L^i \right) = 0.$$

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$$\sum_{i=1}^{I} \left( x_L^{i^*} - \omega_L^i \right) = 0.$$

Competitive equilibrium

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$$u_A(x, y) = x^{\alpha} y^{1-\alpha}$$
  
$$u_B(x, y) = x^{\beta} y^{1-\beta}$$

#### Suppose

$$\alpha = 0.5$$

$$\beta = 0.5$$

$$\omega^{A} = (1.5, 0.5)$$

$$\omega^{B} = (0.5, 1.5)$$

#### Each individual solves

$$\max_{x_i,y_i} \sqrt{x^i y^i}$$

s.t.

$$p_{x}x^{i}+p_{y}y^{i}\leq p_{x}w_{x}^{i}+p_{y}w_{y}^{i}$$

Each individual solves

$$\max_{x_i,y_i} \sqrt{x^i y^i}$$

s.t.

$$p_x x^i + p_y y^i \le p_x w_x^i + p_y w_y^i$$

We can set up a Lagrangean:

$$\mathcal{L} = \sqrt{x^{i}y^{i}} + \lambda \left( p_{x}w_{x}^{i} + p_{y}w_{y}^{i} - p_{x}x^{i} - p_{y}y^{i} \right)$$

Each individual solves

$$\max_{x_i,y_i} \sqrt{x^i y^i}$$

s.t.

$$p_x x^i + p_y y^i \le p_x w_x^i + p_y w_y^i$$

We can set up a Lagrangean:

$$\mathcal{L} = \sqrt{x^i y^i} + \lambda \left( p_x w_x^i + p_y w_y^i - p_x x^i - p_y y^i \right)$$

The FOC are:

$$\frac{1}{2}\sqrt{\frac{y^i}{x^i}} = \lambda p_x$$

$$\frac{1}{2}\sqrt{\frac{x^i}{y^i}} = \lambda p_y$$

Thus,

$$\frac{y^i}{x^i} = \frac{p_x}{p_y}$$
$$y^i = x^i \frac{p_y}{p_y}$$

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We haven't used the budget restriction!

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$$\frac{y^i}{x^i} = \frac{p_x}{p_y}$$
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We haven't used the budget restriction!

$$p_{x}x^{i} + p_{y}y^{i} = p_{x}w_{x}^{i} + p_{y}w_{y}^{i}$$

$$p_{x}x^{i} + p_{y}x^{i}\frac{p_{x}}{p_{y}} = p_{x}w_{x}^{i} + p_{y}w_{y}^{i}$$

$$x^{i} = \frac{w_{x}^{i}p_{x} + w_{y}^{i}p_{y}}{2p_{x}}$$

$$y^{i} = \frac{w_{x}^{i}p_{x} + w_{y}^{i}p_{y}}{2p_{y}}$$

$$x^{A} = \frac{1.5p_{x} + 0.5p_{y}}{2p_{x}}$$

$$y^{A} = \frac{1.5p_{x} + 0.5p_{y}}{2p_{y}}$$

$$x^{B} = \frac{0.5p_{x} + 1.5p_{y}}{2p_{x}}$$

$$y^{B} = \frac{0.5p_{x} + 1.5p_{y}}{2p_{y}}$$

Now we can use condition 2 (market clear)

$$x^{A} = \frac{1.5p_{x} + 0.5p_{y}}{2p_{x}}$$

$$y^{A} = \frac{1.5p_{x} + 0.5p_{y}}{2p_{y}}$$

$$x^{B} = \frac{0.5p_{x} + 1.5p_{y}}{2p_{x}}$$

$$y^{B} = \frac{0.5p_{x} + 1.5p_{y}}{2p_{y}}$$

Now we can use condition 2 (market clear)

$$x^A + x^B = 2$$
$$v^A + v^B = 2$$

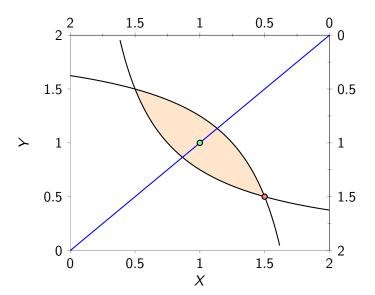
$$\frac{1.5p_x + 0.5p_y}{2p_x} + \frac{0.5p_x + 1.5p_y}{2p_x} = 2$$

$$\frac{p_x}{p_y} = 1$$

$$\frac{1.5p_x + 0.5p_y}{2p_x} + \frac{0.5p_x + 1.5p_y}{2p_x} = 2$$

$$\frac{p_x}{p_y} = 1$$

$$x^A = x^B = y^A = y^B = 1$$



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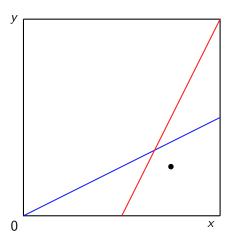
Examples: Cobb-Douglas

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### Suppose that

$$u_A(x^A, y^A) = \min(x^A, 2y^A)$$
$$u_B(x^B, y^B) = \min(2x^B, y^B)$$
$$\omega^A = (3, 1)$$
$$\omega^B = (1, 3)$$



At a given price vector, consumer A can buy any combination  $(x^A, y^A)$  such that:

$$p_{x}w_{x}^{A}+p_{y}w_{y}^{A}\geq p_{x}x^{A}+p_{y}y^{A}$$

At a given price vector, consumer A can buy any combination  $(x^A, y^A)$  such that:

$$p_x w_x^A + p_y w_y^A \ge p_x x^A + p_y y^A$$

or equivalently

$$y^{A} \leq \frac{p_{x}w_{x}^{A} + p_{y}w_{y}^{A}}{p_{y}} - \frac{p_{x}}{p_{y}}x^{A}$$

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$$p_x w_x^A + p_y w_y^A \ge p_x x^A + p_y y^A$$

or equivalently

$$y^{A} \leq \frac{p_{x}w_{x}^{A} + p_{y}w_{y}^{A}}{p_{y}} - \frac{p_{x}}{p_{y}}x^{A}$$

How does this looks in the Edgeworth box?

If  $\frac{p_x}{p_y} \neq 1$  Then, we will have the following restriction:

$$y^A \le \frac{p_x}{p_y} \left( w_x^A - x^A \right) + w_y^A$$

If  $\frac{p_x}{p_y} \neq 1$  Then, we will have the following restriction:

$$y^A \le \frac{p_x}{p_y} \left( w_x^A - x^A \right) + w_y^A$$

Thus, replacing the values of  $w_x^A$  and  $w_y^A$ , we have:

$$y^A \le \frac{p_x}{p_y} \left( 3 - x^A \right) + 1$$

If  $\frac{p_x}{p_y} \neq 1$  Then, we will have the following restriction:

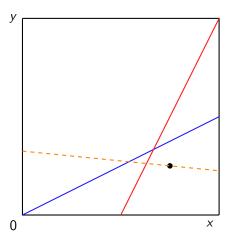
$$y^A \le \frac{p_x}{p_y} \left( w_x^A - x^A \right) + w_y^A$$

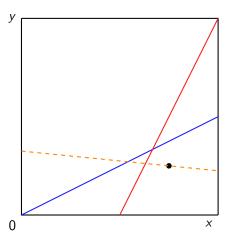
Thus, replacing the values of  $w_x^A$  and  $w_y^A$ , we have:

$$y^A \le \frac{p_x}{p_y} \left( 3 - x^A \right) + 1$$

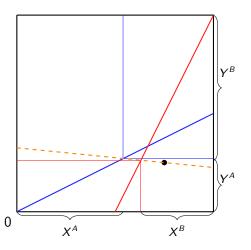
Note that, for the case  $\frac{p_x}{p_y} = 1$ , we have the following restriction:

$$y^A \le 4 - x^A$$

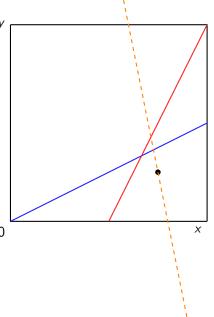


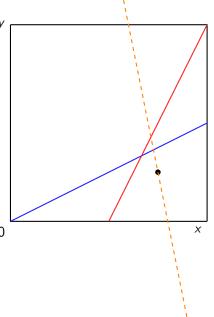


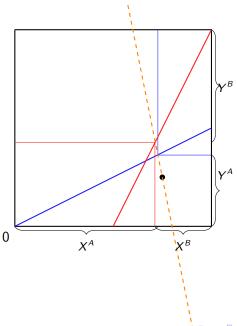
A can buy whats below the orange line, B what is above

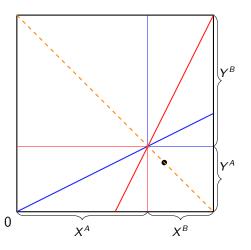


Excess demand of Y and excess supply of X



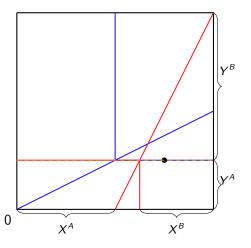




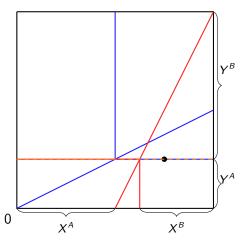


No excess demand or supply

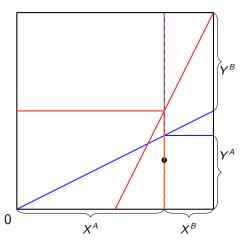
What about zero prices?



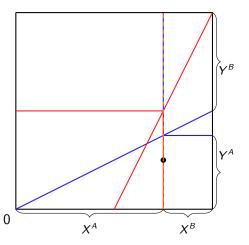
Excess supply of X? (and Y balanced?)



Excess supply of X? (and Y balanced?) Not really since both A and B are indifferent over a wide range that would make the market clear



Excess supply of Y? (and X balanced?)



Excess supply of Y? (and X balanced?) Not really since both A and B are indifferent over a wide range that would make the market clear

#### To sum up...

- ► There are multiple equilibria
- ▶ There are three price vectors associated with these equilibria
- One price vector has a unique resource allocation associated with it
- ▶ Two price vectors ( $p_x = 0$  and  $p_y = 0$ ) have *infinity* resource allocations associated with them

# Perfect complements

Try at home:

$$u_A(x^A, y^A) = \min(x^A, y^A)$$
$$u_B(x^B, y^B) = \min(x^B, y^B)$$
$$\omega^A = (1, 1)$$
$$\omega^B = (3, 1)$$

## Lecture 3: General Equilibrium

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# Lecture 3: General Equilibrium

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Examples: Perfect Substitutes

$$u_A(x^A, y^A) = 2x^A + y^A$$
  

$$u_B(x^B, y^B) = x^B + y^B$$
  

$$\omega^A = (1, 1)$$
  

$$\omega^B = (1, 1)$$

$$u_A(x^A, y^A) = 2x^A + y^A$$
  

$$u_B(x^B, y^B) = x^B + y^B$$
  

$$\omega^A = (1, 1)$$
  

$$\omega^B = (1, 1)$$

 $p_x > 0$  and  $p_y > 0$ , why?

$$u_A(x^A, y^A) = 2x^A + y^A$$
  

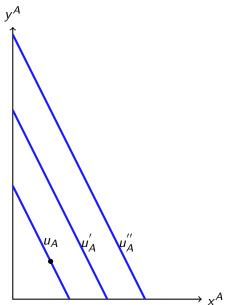
$$u_B(x^B, y^B) = x^B + y^B$$
  

$$\omega^A = (1, 1)$$
  

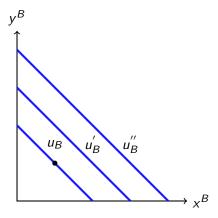
$$\omega^B = (1, 1)$$

 $p_{x}>0$  and  $p_{y}>0$ , why? hence, normalize  $p_{x}=1$ 

Perfect Substitutes Peferences of person A:



Peferences of person B:



## **Algebraic solution**

$$\max_{x^A, y^A} 2x^A + y^A$$

subject to:

$$I = x^{A} + p_{y}y^{A}$$
$$y^{A} \ge 0$$
$$x^{A} \ge 0$$

## Algebraic solution

$$\max_{x^A, y^A} 2x^A + y^A$$

subject to:

$$I = x^{A} + p_{y}y^{A}$$
$$y^{A} \ge 0$$
$$x^{A} > 0$$

From the budget constraint we can obtain  $y^A = \frac{I - x^A}{\rho_y}$ , and adding the condition  $y^A \ge 0$ , we can conclude that  $x^A \in [0, I]$ .

Introducing  $y^A$  into the original maximization problem:

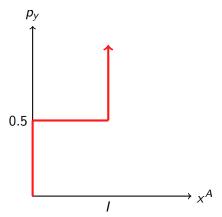
$$\max(2 - \frac{1}{p_y})x^A + \frac{I}{p_y}$$
 s.t.  $x^A \in [0, I]$ 

Which is a maximization of a straight line with slope  $\left(2 - \frac{1}{p_y}\right)$  over an interval.

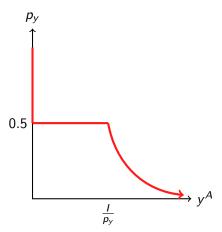
The demand for goods of individual A is

$$X^{A} = \begin{cases} 0 & \text{if } p_{y} < 0.5\\ [0, I] & \text{if } p_{y} = 0.5\\ I & \text{if } p_{y} > 0.5 \end{cases}$$
$$Y^{A} = \begin{cases} \frac{I}{p_{y}} & \text{if } p_{y} < 0.5\\ [0, \frac{I}{p_{y}}] & \text{if } p_{y} = 0.5\\ 0 & \text{if } p_{y} > 0.5 \end{cases}$$

The demand for  $x^A$  is represented below:



The demand for  $y^A$  is represented below:



## Algebraic solution

For person B the solution is analogous, but we have the following maximization problem: Introducing  $y^A$  into the original maximization problem:

$$\max(1 - \frac{1}{p_y})x^B + \frac{I}{p_y}$$
 s.t.  $x^B \in [0, I]$ 

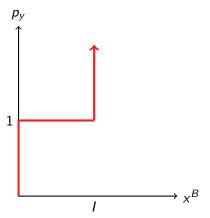
Which is a maximization of a straight line with slope  $\left(1 - \frac{1}{\rho_y}\right)$  over an interval.

The demand for goods of individual B is

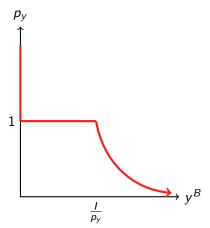
$$X^{B} = \begin{cases} 0 & \text{if } p_{y} < 1\\ [0, I] & \text{if } p_{y} = 1\\ I & \text{if } p_{y} > 1 \end{cases}$$

$$Y^{B} = \begin{cases} \frac{I}{p_{y}} & \text{if } p_{y} < 1\\ [0, \frac{I}{p_{y}}] & \text{if } p_{y} = 1\\ 0 & \text{if } p_{y} > 1 \end{cases}$$

The demand for  $x^B$  is represented below:



The demand for  $y^B$  is represented below:



# Perfect Substitutes When is the market for good *X* balanced (how about good *y*?)

When is the market for good X balanced (how about good y?)

▶ Try  $p_y < 0.5$ 

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$

- ► Try  $p_y < 0.5$
- $X^A = 0 \text{ and } X^B = 0$
- ▶ Try  $p_y = 0.5$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- ▶ Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- Try  $0.5 < p_y < 1$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ► Try  $p_V = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- ▶ Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- ► Try  $0.5 < p_v < 1$
- $X^A = I$  and  $X^B = 0$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- ▶ Can't be an equilibrium since I = 1.5 when  $p_y = 0.5$ , thus  $X^A + X^B < 2$
- ► Try  $0.5 < p_v < 1$
- $X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_y$ , thus  $X^A + X^B < 2$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- ► Try  $0.5 < p_v < 1$
- $X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_V$ , thus  $X^A + X^B < 2$
- $Try p_v = 1$
- $X^A = I = 2$  and  $X^B = [0, 2]$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- ► Try  $0.5 < p_v < 1$
- $X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_y$ , thus  $X^A + X^B < 2$
- $Try p_y = 1$
- $X^A = I = 2$  and  $X^B = [0, 2]$
- ▶ One possible equilibrium  $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$

- ► Try  $p_y < 0.5$
- $X^A = 0$  and  $X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- ► Try  $0.5 < p_{V} < 1$
- $X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_y$ , thus  $X^A + X^B < 2$
- ightharpoonup Try  $p_{v}=1$
- $X^A = I = 2$  and  $X^B = [0, 2]$
- ▶ One possible equilibrium  $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$

- ► Try  $p_y < 0.5$
- $X^A = 0 \text{ and } X^B = 0$
- ▶ Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- Can't be an equilibrium since I=1.5 when  $p_y=0.5$ , thus  $X^A+X^B<2$
- ► Try  $0.5 < p_v < 1$
- $X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_y$ , thus  $X^A + X^B < 2$
- $Try p_v = 1$
- $X^A = I = 2$  and  $X^B = [0, 2]$
- ▶ One possible equilibrium  $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$
- $ightharpoonup Try p_y > 1$
- $X^A = I$  and  $X^B = I$

- ▶ Try  $p_y < 0.5$ 
  - $X^A = 0 \text{ and } X^B = 0$
  - ► Try  $p_y = 0.5$
- $X^A = [0, I] \text{ and } X^B = 0$
- ► Can't be an equilibrium since I = 1.5 when  $p_y = 0.5$ , thus  $X^A + X^B < 2$
- ► Try  $0.5 < p_v < 1$
- $\triangleright X^A = I$  and  $X^B = 0$
- ▶ Can't be an equilibrium since  $I = 1 + p_v$ , thus  $X^A + X^B < 2$
- ightharpoonup Try  $p_{v}=1$
- $X^A = I = 2$  and  $X^B = [0, 2]$
- ▶ One possible equilibrium  $(X^A = 2, X^B = 0, Y^A = 0, Y^B = 2)$
- $Try p_v > 1$
- $X^A = I$  and  $X^B = I$
- $X^A = I = 1 + p_V$  and  $X^B = I = 1 + p_V$
- Can't be an equilibrium since  $I = 1 + p_y$ , thus  $X^A + X^B = 2 + 2p_y > 2$